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ON RANDOM DETERMINANTS I

András Prékopa (Budapest)

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1 Introduction

Consider the determinant with random entries

$$\Delta_n = \begin{vmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{12} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{vmatrix}, \tag{1.1}$$

where we suppose that the random variables ξ_{ik}, $i,k = 1,\dots,n$ are independent and identically distributed. (The reader will observe that certain conditions can be weakened without violating the validity of our subsequent statements.) We shall assume later on the existence of the moments of the ξ_{ik}'s of order as high as it will be necessary. We are now interested in finding the moments $\mathbb{E}(\Delta_n^{2k})$, $k = 1, 2, \dots$. The odd order moments of Δ_n are clearly equal to 0 as Δ_n has a symmetrical distribution with respect to 0. In fact interchanging two rows in Δ_n we obtain $-\Delta_n$ and this latter has the same probability distribution as Δ_n. Suppose that $\mathbb{E}(\xi_{ik}) = 0$, $\mathbb{E}(\xi_{ik}^2) = 1$, $i,k = 1,\dots,n$. Then it is well known that[1] (first remarked in [4])

$$\mathbb{D}^2(\Delta_n) = \mathbb{E}(\Delta_n^2) = n!. \tag{1.2}$$

The fourth moment of Δ_n was obtained by Nyquist, Rice and Riordan [6] and the formula is the following

$$\mathbb{E}(\Delta_n^4) = \frac{(n!)^2}{2} \sum_{i=0}^{n} \frac{(n-i+1)(n-i+2)}{i!}(m_4 - 3)^i, \tag{1.3}$$

where

$$m_4 = \mathbb{E}(\xi_{ik}^4), \qquad i,k = 1,\dots,n.$$

In an earlier paper Turán and Szekeres [1] (see also [2], [3]) investigated the sum of squares and the sum of the fourth powers of all determinants with entries -1, 1. Applying

[1] $\mathbb{E}(\xi)$ denotes the expectation and $D(\xi)$ the dispersion of the random variable ξ.

non-probabilistic arguments they obtained the formula (1.2) for the arithmetic mean of all squares and a recursion formula for the arithmetic mean of the fourth powers. This recursion formula was not solved, however, but it is a special case of the recursion formula proved later in [6] for $\mathbb{E}(\Delta_n^4)$ which lead to (1.3). We can therefore obtain from (1.3) the explicit formula for the arithmetic mean of all fourth powers of the determinants with entries -1, 1 if we substitute $m_4 = 1$ in (1.3).

There is only one type of probability distributions as the distribution of the ξ_{ik}'s for which all moments of Δ_n are known and this is the standard normal distribution. In this case we have

$$\mathbb{E}(\Delta_n^{2k}) = n!\frac{(n+2)!}{2!}\frac{(n+4)!}{4!}\dots\frac{(n+2k-2)!}{(2k-2)!}. \tag{1.4}$$

This result can be obtained in a well known way from the Wishart distribution (see e.g. [7]). Other proof is published in [6]. In a summary of a lecture Forsyth and Tukey [5] gave without proof a formula for the $2k$-th moment of the content of n random unit vectors uniformly distributed on the surface of the unit sphere in the n-dimensional space. Formula (1.4) can simply be obtained from this and vice versa. The proof was never published. We shall give a direct proof for that without using any deeper tools as this case seems to be of particular interest.

2 Reformulation of the Problem. Moments of Permanents. New Proof of an Earlier Result

Together with the random determinant

$$\Delta_n = \sum_{(i_1,i_2,\dots,i_n)} \pm\xi_{1i_1},\xi_{2i_2}\dots\xi_{ni_n}$$

we shall investigate the random permanent

$$P_n = \sum_{(i_1,i_2,\dots,i_n)} \xi_{1i_1},\xi_{2i_2}\dots\xi_{ni_n}$$

of the same random matrix. We assume that the random variables ξ_{ij} are symmetrically distributed with respect to 0. Let us introduce the notation $m_{2k} = \mathbb{E}(\xi_{ij}^{2k})$. The problem of finding the $2k$-th moment of the random variable P_n can be reformulated in the following way. Consider all tables of $2k$ rows and n columns one row of which consists of a permutation of the elements $1, 2, \dots, n$. The number of all such tables is $(n!)^{2k}$. A table is called regular if every number in every column has an even multiplicity. We assign a weight to each column and define the weight of a regular table as the product of the weights of the columns. The weight of a column is defined as

$$m_2^{j_1} m_4^{j_2} \dots m_{2k}^{j_k}, \qquad m_2 = 1,$$

where $2j_1+4j_2+\dots+2kj_k = 2k$ and j_1 is the number of different numbers with multiplicity 2, j_2 is the number of different numbers with multiplicity 4 in that column etc. If at least one column contains a number with an odd multiplicity then the weight of the table is 0 by definition. The sum of the weights of the tables is equal to $\mathbb{E}(P_n^{2k})$.

Let us now give a positive or a negative sign to each table according that the sum of inversions contained in the different rows is an even or an odd number. The sum of the signed weights is equal to $\mathbb{E}(\Delta_n^{2k})$.

The above assertions follow immediately from the definition of the permanent and the determinant taking into account the independence of the random variables $\xi_{ik},\ i,k = 1,\dots,n$. We observe that P_n has also a symmetrical distribution with respect to 0. Now we prove the following

THEOREM 1 $\mathbb{E}(P_n^2) = \mathbb{E}(\Delta_n^2)$, $\mathbb{E}(P_n^4) = \mathbb{E}(\Delta_n^4)$, $n = 1, 2, \dots,$

$$\mathbb{E}(P_n^{2k}) = \mathbb{E}(\Delta_n^{2k}), \quad \textit{for} \quad n = 1, 2, ;\ k = 1, 2, \dots,$$

but if $\mathbb{P}(\xi_{ij} = 0) \neq 1$ *then*

$$\mathbb{E}(P_n^{2k}) \neq \mathbb{E}(\Delta_n^{2k}) \quad \textit{for} \quad k \geq 3;\ n \geq 3.$$

PROOF. The validity of the first equality is trivial. When proving the second one we give at the same time a proof for the formula (1.3). We shall make use of the above reformulation of the moment-problem and consider all $4 \times n$ tables where each number in each column has an even multiplicity. Any multiplicity can be now just either 2 or 4. We may fix the permutation of the first row as $1\ 2\ 3 \dots n$ and at the end multiply the result by $n!$

Consider together the first and the second rows:

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & n, \\ j_1 & j_2 & j_3 & \dots & j_n. \end{array}$$

This is conceivable as one permutation. Let $i_1, i_2, \dots, i_n$ denote the number of cycles of lengths $1, 2, \dots, n$, respectively. The $4 \times n$ table is regular if and only if in the third and fourth rows below each cycle with the same numbers in the same ordering is repeated what stands in the first and second rows but there are two possibilities. Below from the considered cycle the third row may contain the above standing part i.e. the first row while the fourth row contains the corresponding part of the second row or conversely.

These two possibilities can be used independently of each other below each cycle of the permutation defined by the first two rows. To illustrate the situation consider the first three numbers of the first and second rows, and suppose that they form the following cycle:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1. \end{array}$$

Now in the first column we must have one more 1 and one more 2 to obtain a regular table. This can be done so that 1 stands in the third row and 2 in the fourths or conversely. This choice uniquely determines the other two elements in the third row and also in the fourth row. Therefore we have

$$\text{either}\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{array} \qquad \text{or} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3. \end{array}$$

Having this structure of the $4 \times n$ regular tables we show that every such table has a positive sign. Applying a permutation for the n columns so that elements in the cycles in the first two rows be connected and stand after each other, a regular table keeps the weight and the sign. In this new table from the point of view of the weight and sign it is immaterial whether the elements inside a cycle in the second row are repeated in the third or in the fourth row. In fact a cycle of an odd length has an even number of transpositions and a cycle of an even length has an odd number of transpositions therefore the internal number of transpositions remains the same. The external number of transpositions also remains the same thus the new table is not sensitive for such a change from the point of view of signed weight. But if the whole first row is placed in the third and the whole second row is placed in the fourth row then the table is clearly positive. This proves the second assertion of the theorem.

To prove that $\mathbb{E}(P_n^{2k}) \neq \mathbb{E}(\Delta_n^{2k})$ if $m_2 \neq 0$, $n \geq 3$, $k \geq 3$, it is enough to show that there are tables with negative weights. If $n = 3$ and $k = 3$ then the signed weight of the table

$$\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{array}$$

is $-m_2^9 < 0$. For arbitrary $n \geq 3$ and $k \geq 3$ it suffices to supply the above table by adding $4\ 5\ 6\ \ldots\ n$ to each row in the same permutation and adding as many new rows as it is necessary containing the same permutations. The obtained table is surely negative. Finally it is easy to see that

$$\mathbb{E}(P_n^{2k}) = \mathbb{E}(\Delta_n^{2k}) \qquad \text{for} \qquad n = 1, 2;\ k = 1, 2, \ldots.$$

To derive formula (1.3) we remark that the columns of a regular table can be subdivided into three categories. The first category contains columns which have four times the same number. The second one contains columns which have the same numbers in the first and second rows also in the third and fourth rows but these numbers are different. The remaining columns belong to cycles of length at least 2 of the first two rows and these columns form the third category. In order to obtain the number of tables (which is the sum of weights in this case) all columns of which belong to the third category we mention that if $d(n, k)$ is the number of permutations consisting of k cycles with lengths at least 2 then it is well known that (see e.g. [8])

$$d_n(t) = \sum_{k=0}^{n} d(n,k)t^k = \sum_{k=0}^{n} \binom{n}{k} t(t+1)\dots(t+n-k-1)(-t)^k.$$

As each cycle can be repeated either in the third or in the fourth row, $d_n(2)$ gives the number of tables having columns belonging just into the third category.

If a table consists of columns of the second kind then the first two rows are identical and also the third and fourth rows. As there is no column containing four times the same number, the number of all $4 \times n$ tables is the sum of weights and it is $d_n(1)$. Thus

$$\begin{aligned}\mathbb{E}(\Delta_n^4) &= n! \sum_{i_1+i_2+i_3=n} \frac{n!}{i_1!i_2!i_3!} d_{i_1}(1) d_{i_2}(2) m_4^{i_3} \\ &= \frac{(n!)^2}{2} \sum_{k=0}^{n} \frac{(n-k+1)(n-k+2)}{k!} (m_4-3)^k.\end{aligned}$$

We remark that the numbers $d(n,k)$ are associated Stirling numbers of the first kind.

3 The Case of the Standard Normal Distribution

We suppose that the ξ_{ij}'s in (1.1) have standard normal distribution and give a proof for the formula (1.4). For this purpose first we prove the following

LEMMA. *Let $\zeta_1, \dots, \zeta_k$ be n-dimensional independent random vectors with independent components having standard normal distribution. The k-dimensional content of the parallelotope determined by these vectors is the product of two independent random variables one of which has a χ-distribution with $n-k+1$ degrees of freedom and the other is distributed as the $k-1$-dimensional content of $k-1$ independent random vectors having independent and standard normally distributed components.*

PROOF. Let $\Delta_n^{(k)}$ denote the content of the k random vectors. Then

$$\Delta_n^{(k)} = \alpha_k \Delta_n^{(k-1)},$$

where $\Delta_n^{(k-1)}$ is the $k-1$-dimensional content of the parallelotope determined by $\xi_1, \dots,$ ξ_{k-1} and α_k is the distance of ζ_k from the subspace spanned by $\zeta_1, \dots, \zeta_{k-1}$. In view of the spherical symmetry of the distribution of ζ_i, α_k and $\Delta_n^{(k-1)}$ are independent of each other. α_k is clearly a χ-variable with $n-k+1$-degrees of freedom as the subspace of the first $k-1$ vectors can be fixed as the set of those points $(x_1, x_2, \dots, x_n)$ for which $x_k = x_{k+1} = \dots = x_n = 0$. This completes the proof. □

THEOREM 2 *If the ξ_{ij}'s have standard normal distribution then the random variable* (1.1) *can be written as the product of n independent χ-variables:*

$$\Delta_n = \chi_1 \chi_2 \cdots \chi_n,$$

where α_k has $n-k+1$ degrees of freedom.

Proof. The theorem follows from a subsequent application of the idea of the proof in the preceding Lemma.

As the k-th moment of a χ^2-variable with i-degrees of freedom is equal to

$$(i+2k-2)(i+2k-4)\dots(i+2)i,$$

it follows that

$$\mathbb{E}(\Delta_n^{2k}) = \prod_{i=1}^{n}(i+2k-2)(i+2k-4)\dots(i+2)i = n!\frac{(n+2)!}{2!}\frac{(n+4)!}{4!}\dots\frac{(n+2k-2)!}{(2k-2)!}$$

which proves (1.4). We remark that the moments of the content of n random vectors uniformly distributed on the surface of the unit sphere in the n-dimensional space can be obtained from this because

$$\Delta_n = \chi_1\chi_2\dots\chi_n \begin{vmatrix} \frac{\xi_{11}}{\chi_1} & \dots & \frac{\xi_{1n}}{\chi_n} \\ \dots & \dots & \dots \\ \frac{\xi_{n1}}{\chi_1} & \dots & \frac{\xi_{nn}}{\chi_n} \end{vmatrix},$$

where

$$\chi_i = \sqrt{\xi_{i1}^2 + \dots + \xi_{in}^2}, \qquad i = 1, \dots, n$$

and the $n+1$ factors in the product as well as the rows of the determinant are independent.

□

4 Polynomials Associated with Random Determinants, Generalization of the Formula (1.3)

Let us define the polynomials $f_n(m_1, m_2, \dots, m_k)$, $k, n = 1, 2, \dots$ as the sum signed weights of all $k \times n$ tables where in each row we write one permutation of the numbers $1, 2, \dots, n$, the weight of a table is the product of the weights of the columns and the weight of a column is $m_1^{i_1} m_2^{i_2} \dots m_k^{i_k}$ where i_j is the number of different numbers with multiplicity j in the column. The sign is the total sum of the transpositions in the k rows. The non-signed sum of weights will be denoted by $g_n(m_1, m_2, \dots, m_k)$. The variables $m_1, m_2, \dots, m_k$ can be real or complex. Considering a random determinant (1.1) where the random entries are independent, identically (but not necessarily symmetrically) distributed having finite moments up to order k, and these moments are $m_1, m_2, \dots, m_k$, then

$$f_n(m_1, m_2, \dots, m_k) = \mathbb{E}(\Delta_n^k), \tag{4.1}$$

while

$$g_n(m_1, m_2, \dots, m_k) = \mathbb{E}(P_n^k). \tag{4.2}$$

As Δ_n has a symmetrical distribution with respect to 0, $f_n(m_1, m_2, \ldots, m_k)$ vanishes if $m_1, m_2, \ldots, m_k$ are moments of a probability distribution and k is odd. This implies that $f_n(m_1, m_2, \ldots, m_k)$ vanishes for all values of the variables $m_1, m_2, \ldots, m_k$ if k is an odd number. The same holds for g_n if the entries have symmetrical distribution with respect to 0. The polynomials f_n, g_n will be called polynomials associated with random determinants, random permanents, respectively. Both f_n and g_n are clearly homogeneous polynomials of their variables. We mention also the following

THEOREM 3 *For fixed k and $m_1, m_2, \ldots, m_{k-1}$, the polynomials $g_n/n!$ are Appel polynomials of the variable m_k. The same holds for $f_n/n!$ if k is an even number.*

PROOF. Note that polynomials $y_1(x), y_2(x), \ldots$ are called Appel polynomials if $y_n'(x) = ny_{n-1}(x)$, $n = 1, 2, \ldots$. To prove this property of the above polynomials consider the $k \times n$ tables. Each table contains a certain number of columns consisting of k times the same number. If the number of such columns is j then they can be selected in $\binom{n}{j}$ different ways. Thus g_n has the form

$$g_n(m_1, m_2, \ldots, m_k) = n! \sum_{j=0}^{n} \binom{n}{j} m_k^j d_{n-j}(m_1, \ldots, m_{k-1}). \tag{4.3}$$

f_n has a similar form but we have to remark that if k is even then any particular choice of the j columns consisting of k times the same numbers the remaining columns form a $k \times (n-j)$ table of the same sign. Thus

$$f_n(m_1, m_2, \ldots, m_k) = n! \sum_{j=0}^{n} \binom{n}{j} m_k^j c_{n-j}(m_1, \ldots, m_{k-1}). \tag{4.4}$$

Our assertions follow immediately from (4.3) and (4.4).

If the random variables ξ_{ij} in (1.1) have a symmetrical distribution then $m_1 = m_3 = m_5 = \ldots = 0$. If moreover we take into account that $m_2 = 1$ then we have polynomials $g_n(m_4, \ldots, m_{2k}), f_n(m_4, \ldots, m_{2k})$. Now we generalize the formula (1.3) and express it in

THEOREM 4 *If the random variables ξ_{ij} in (1.1) have a symmetrical distribution and this has a finite moment of order $2k$ moreover the moments of order $2, 4, \ldots, 2k-2$ are the same as those of the standard normal distribution,*

$$m_{2j} = \frac{(2j)!}{j!2^j}, \qquad j = 1, 2, \ldots, k-1 \tag{4.5}$$

while m_{2k} is arbitrary, then

$$\mathbb{E}(\Delta_n^{2k}) = (n!)^2 \sum_{j=0}^{n} \frac{1}{j!} \left(m_{2k} - \frac{(2k)!}{k!2^k} \right)^j \frac{M_{n-j}^{(2k)}}{[(n-j)!]^2}, \tag{4.6}$$

where $M_n^{(2k)}$ stands for the $2k$-th moment of Δ_n the entries of which have the standard normal distribution, i.e. $M_n^{(2k)}$ is given by (1.4).

PROOF. From Theorem 3 we know that

$$\frac{d}{dm_{2k}} \frac{\mathbb{E}(\Delta_n^{2k})}{n!} = n \frac{\mathbb{E}(\Delta_{n-1}^{(2k)})}{(n-1)!} \tag{4.7}$$

where we have the initial conditions

$$\mathbb{E}(\Delta_n^{2k}) = M_n^{(2k)} \qquad \text{for} \qquad m_{2k} = \frac{(2k)!}{2^k k!}. \tag{4.8}$$

The sequence of polynomials $\mathbb{E}(\Delta_1^{2k}), \mathbb{E}(\Delta_2^{2k}), \ldots$ is uniquely determined by (4.7) and (4.8). But (4.6) satisfies these conditions hence our theorem is proved. □

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