MIDTERM 2 (4 problems)

Problem 1 (30 points). Set up a LP problem for the following situations.

An oil refinery produces four type of raw gasoline: alkylate, catalytic-cracked, straight-run and isopentane. Two important characteristics of each gasoline are its performance number (PN) and its vapor pressure (RVP). These two characteristics together with production levels in barrel per day, are as follows:

<table>
<thead>
<tr>
<th></th>
<th>PN</th>
<th>RVP</th>
<th>Barrels produced</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alkylate</td>
<td>107</td>
<td>5</td>
<td>3814</td>
</tr>
<tr>
<td>Catalytic-cracked</td>
<td>93</td>
<td>8</td>
<td>2666</td>
</tr>
<tr>
<td>Straight-run</td>
<td>87</td>
<td>4</td>
<td>4016</td>
</tr>
<tr>
<td>Isopentane</td>
<td>108</td>
<td>21</td>
<td>1300</td>
</tr>
</tbody>
</table>

These gasolines can be sold either raw, at $4.83 per barrel, or blended into aviation gasoline “Avgas A” and “Avgas B”. Quality standards impose certain requirements on aviation gasolines; these requirements, together with the selling prices, are as follows:

<table>
<thead>
<tr>
<th></th>
<th>PN</th>
<th>RVP</th>
<th>Price per barrel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avgas A</td>
<td>at least 100</td>
<td>at most 7</td>
<td>$6.45</td>
</tr>
<tr>
<td>Avgas B</td>
<td>at least 91</td>
<td>at most 7</td>
<td>$5.91</td>
</tr>
</tbody>
</table>

The PN and RVP of each mixture are simply weighted averages of the PNs and RVPs of its components. Set up a linear program to maximize the profit.

Solution.

I will use the following notations.

Constants:

\[ P_1, \cdots, P_4 \] PN for different types of raw gasoline;

\[ R_1, \cdots, R_4 \] RVP for different types of raw gasoline;

\[ Q_1, \cdots, Q_4 \] Produced volume of different types of raw gasoline;

\[ C_0, C_A, C_B \] Price of raw gasoline, Avgas A and Avgas B;

\[ P_A, P_B \] Minimal PN for Avgas;

\[ R_A, R_B \] Maximal RVP for Avgas;

Variables:

\[ x_{A1}, \cdots, x_{B4} \] Volume of gasoline going to corresponding Avgas.
Then our problem looks

Maximize \((C_A - C_0) \sum_{i=1}^{4} x_{Ai} + (C_B - C_0) \sum_{i=1}^{4} x_{Bi} + C_0 \sum_{i=1}^{4} Q_i\)

subject to

\[\begin{align*}
& x_{A1} + x_{B1} \leq Q_1 \\
& x_{A2} + x_{B2} \leq Q_2 \\
& x_{A3} + x_{B3} \leq Q_3 \\
& x_{A4} + x_{B4} \leq Q_4 \\
& \sum_{i=1}^{4} x_{Ai}(P_i - P_A) \geq 0 \\
& \sum_{i=1}^{4} x_{Bi}(P_i - P_B) \geq 0 \\
& \sum_{i=1}^{4} x_{Ai}(R_i - R_A) \leq 0 \\
& \sum_{i=1}^{4} x_{Bi}(R_i - R_B) \leq 0
\end{align*}\]

Problem 2 (30 points). Solve the problem, then write the optimality conditions and check them.

Maximize \(-2x_1 - x_2 + x_3 + 2x_4 + x_5\)

subject to

\[\begin{align*}
& x_1 + x_2 + 2x_3 + x_4 + x_5 = 8 \\
& x_1 - x_2 + x_3 + 3x_4 - 2x_5 = -2 \\
& 0 \leq x_1 \leq 5 \\
& 0 \leq x_2 \leq 10 \\
& 0 \leq x_3 \leq 2 \\
& 0 \leq x_4 \leq 3 \\
& 0 \leq x_5 \leq 2
\end{align*}\]

Solution.

Again we can take initial basis \(x_1, x_2\):

\[\begin{align*}
B &= x_1, x_2 \\
x_B &= (3, 5)^T \\
z &= -11 \\
B^{-1} &= \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} \\
c_B &= (-2, -1) \\
y &= (-1.5, -0.5) \\
c' &= (0, 0, 4.5, 5, 1.5)
\end{align*}\]
Can choose any. Chosen: \( x_4 \).

\[
d = (2, -1)^T
\]

. \( x_1 = 0 \) leaves.

\[
B = x_4, x_2
\]

\[
x_B = (1.5, 6.5)^T
\]

\[
z = -3.5
\]

\[
B^{-1} = \begin{pmatrix} 0.25 & 0.25 \\ 0.75 & -0.25 \end{pmatrix}
\]

\[
c_B = (2, -1)
\]

\[
y = (-0.25, 0.75)
\]

\[
c' = (-2.5, 0, 0.75, 0, 2.75)
\]

Can choose \( x_3, x_5 \). Chosen: \( x_5 \).

\[
d = (-0.25, 1.25)^T
\]

. \( x_5 \) changes value to 2 and does not enter the basis.

\[
x_B = (2, 4)^T
\]

\[
z = 2
\]

Can choose \( x_3 \).

\[
d = (0.75, 1.25)^T
\]

. \( x_3 \) changes value to 2 and does not enter the basis.

\[
x_B = (0.5, 1.5)^T
\]

\[
z = 3.5
\]

Optimal.

\[
x_1 = 0, x_2 = 1.5, x_3 = x_5 = 2, x_4 = 0.5, z = 3.5
\]

Optimality conditions.

\[
\begin{align*}
\text{Minimize} & \quad 8v_1 - 2v_2 + 5y_1 + 10y_2 + 2y_3 + 3y_4 + 2y_5 \\
\text{subject to} & \quad v_1 + v_2 + y_1 \geq -2 \\
& \quad v_1 - v_2 + y_2 \geq -1 \\
& \quad 2v_1 + v_2 + y_3 \geq 1 \\
& \quad v_1 + 3v_2 + y_4 \geq 2 \\
& \quad v_1 - 2v_2 + y_5 \geq 1 \\
& \quad y_1, ..., y_5 \geq 0
\end{align*}
\]

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The theorem says that in the optimal solution \( y_1 = y_2 = y_4 = 0 \) and the conditions for \( x_2, \ldots, x_5 \) must be satisfied as equalities. So we get:

\[
\begin{align*}
v_1 + v_2 &\geq -2 \\
v_1 - v_2 &\geq -1 \\
2v_1 + v_2 + y_3 &\geq 1 \\
v_1 + 3v_2 &\geq 2 \\
v_1 - 2v_2 + y_5 &\geq 1 \\
y_3, y_5 &\geq 0
\end{align*}
\]

From the second and the fourth line we get \( v_2 = 0.75, v_1 = -0.25 \). The first line is satisfied. From the third and the fifth line we get \( y_3 = 0.75, y_5 = 2.75 \). We can also check that the optimal value is the same.

Problem 3 (15 points). Construct the dual problems for

\[\text{Maximize } x_1 - x_2 \]
subject to
\[
\begin{align*}
2x_1 + 3x_2 - x_3 + x_4 &\leq 0 \\
3x_1 + x_2 + 4x_3 - 2x_4 &\geq 3 \\
-x_1 - x_2 + 2x_3 + x_4 &\leq 1
\end{align*}
\]

and for the problem (the constraints are the same)

\[\text{Minimize } x_1 - x_2 \]
subject to
\[
\begin{align*}
2x_1 + 3x_2 - x_3 + x_4 &\leq 0 \\
3x_1 + x_2 + 4x_3 - 2x_4 &\geq 3 \\
-x_1 - x_2 + 2x_3 + x_4 &\geq 1
\end{align*}
\]

Solution.

It was written on the board \( x_2, x_3 \geq 0 \) \( x_1, x_4 \) are free.

First problem.

\[\text{Minimize } 3y_2 + y_3 \]
subject to
\[
\begin{align*}
2y_1 + 3y_2 - y_3 &\geq 1 \\
3y_1 + y_2 - y_3 &\geq -1 \\
y_1 + 4y_2 + 2y_3 &\geq 0 \\
y_1 - 2y_2 + y_3 &\geq 0 \\
y_1 &\geq 0 \\
y_2 &\leq 0 \\
y_3 &\text{ is free}
\end{align*}
\]

Second problem.

\[\text{Maximize } 3y_2 + y_3 \]
subject to
\[
\begin{align*}
2y_1 + 3y_2 - y_3 &\geq 1 \\
3y_1 + y_2 - y_3 &\leq -1 \\
y_1 + 4y_2 + 2y_3 &\leq 0 \\
y_1 - 2y_2 + y_3 &\leq 0 \\
y_1 &\leq 0 \\
y_2 &\geq 0 \\
y_3 &\text{ is free}
\end{align*}
\]
The condition written on the board affects only equality/inequality signs for constraints: equality sign corresponds to free variables, ≥ in minimization problem and ≤ in maximization problem correspond to non-negative ones.

Problem 4 (25 points). In the game with the payoff matrix

\[
\begin{bmatrix}
3 & 2 & 0 & -1 & 5 & -2 \\
-2 & -3 & 2 & 4 & 0 & 4 \\
5 & -3 & 4 & 0 & 4 & 7 \\
1 & 3 & 3 & 2 & -6 & 5
\end{bmatrix}
\]

the row player’s mixed strategy \([\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}]\) is optimal. Describe all the optimal strategies for column (minimizer) player.

Solution.

If we multiply the given vector \([\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}]\) by matrix

\[
\begin{pmatrix}
3 & 2 & 0 & -1 & 5 & -2 \\
-2 & -3 & 2 & 4 & 0 & 4 \\
5 & -3 & 4 & 0 & 4 & 7 \\
1 & 3 & 3 & 2 & -6 & 5
\end{pmatrix}
\]

we get vector \([1.25, 1.25, 1, 1, 1.25]\). It means that the value of the game is 1, that the column player uses only columns 2, 4, 5 and that any optimal against row strategy 1, 2, 4 should give exactly 1.

So, we get the system:

\[
\begin{align*}
y_2 + y_4 + y_5 &= 1 \\
2y_2 - y_1 + 5y_5 &= 1 \\
-3y_2 + 4y_4 &= 1 \\
3y_2 + 2y_4 - 6y_5 &= 1
\end{align*}
\]

We can see, that it has the only solution \([\frac{1}{3}, \frac{1}{2}, \frac{1}{6}]\). So, the only optimal strategy is \([0, \frac{1}{3}, 0, \frac{1}{2}, \frac{1}{6}, 0]\).