Introduction to Probability
MSIS 575
Final Exam

Instructor: Farid Alizadeh
Date: Tuesday December 19, 2000

December 19, 2000

PLEASE READ VERY CAREFULLY BEFORE STARTING TO WORK ON THE EXAM

RULES:

• The exam is on Thursday December 21 from 1 pm to 2:30 pm in class.

• Find the solutions to the questions below; these are going to be the questions that will be given in the exam except those numbers, words or parameters that are boxed may be different in the actual exam on Thursday.

• You may obtain the answers to the questions by any means. However on Thursday the exam will be closed book and closed notes and you have to give the solutions in your own words.

• All your answers must be accompanied by complete, yet succinct and to the point explanations. If your answer is correct but there is not sufficient explanation, or your explanation is nonsensical, you will get no credit!
1. Let $X$ and $Y$ be two independent discrete random variables taking values $0, 1, 2, \ldots$. If $U(t)$ is the generating function for the probability density function of $X$ and $V(t)$ is the generating function for the probability density function of $Y$ prove that the generating function for the probability density of the random variable $Z = X - Y$ is $U(t)V(1/t)$. (Note: $Z$ can take values $0, \pm 1, \pm 2, \ldots$, so its generating function is given by double infinite sum $\cdots + f_{-2}t^{-2} + f_{-1}t^{-1} + f_0 + f_1t + f_2t^2 + \cdots$, where $f_j = \Pr[Z = j]$.)

2. Let $P$ be an ergodic Markov chain (that is all of its states are ergodic) with $n$ states and with stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$. Let $Q$ be an $n \times n$ matrix defined as follows:

$$Q_{ij} = \frac{\pi_j}{\pi_i} p_{ji}$$

(a) Prove that $Q$ is a stochastic matrix and therefore corresponds to some Markov chain. (The chain corresponding to $Q$ is called time reversal of $P$).

(b) Recall that a Markov chain with $n$ states is a sequence of random variables $X_i$ which take values from the set $1, 2, \ldots, n$, and $p_{ij} = \Pr[X_{k+1} = j|X_k = i]$. Using Bay’s formula express $Q_{ij}$ in terms of conditional probabilities of random variables $X_{k+1}$ and $X_k$. (This will justify the term “time reversal”.)

(c) Draw the chain corresponding to the time reversal of the following chain (If sum of probabilities out of a state do not add up to one, assume a loop on that state with transition probability equal to the slack):

(d) Prove that $\pi$ is the stationary distribution of the time reversal of $P$ as well.

(e) A Markov chain is time reversible if $P = Q$. Give an example of a time reversible chain with a single closed set and at least four states.

3. Consider the following joint distribution of two discrete random variable $X$ and $Y$:

\[
\begin{array}{c|cccc}
X \rightarrow & 1 & 2 & 3 & 4 \\
\hline 
Y \downarrow & & & & \\
1 & 1/5 & 1/6 & 1/7 & 1/8 \\
2 & 1/9 & 1/10 & 1/11 & 1/12 \\
3 & 1/13 & 1/14 & 1/15 & 1/16 \\
\end{array}
\]

(a) What are the marginal distribution of random variables $X$ and $Y$?

(b) What is the conditional density function $f(x|Y = 2)$?

(c) Are $X$ and $Y$ independent or not? Explain your answer.

(d) What is the conditional expectation $E(Y|X = 1)$?
4. Let $X$ be chosen at random (that is with uniform distribution) in the interval $(0, 1)$, and denote by $U$ the length of the shorter of the intervals $(0, X)$ and $(X, 1)$. Let $V = 1 - U$ be the length of the longer interval.

(a) What are the distributions of random variables $U$ and $V$? Are they independent? What are the means of $U$ and $V$?

(b) Compute the probability density function and the distribution function of the random variable $Z = \frac{V}{U}$. What is the expected value of $Z$?

(c) What are the joint distribution $F(u, v)$ and the joint density of $f(u, v)$ of $U$ and $V$? Compute the conditional expectation $E(U|V = 0.6)$?