Original Problem
Given a general linear programming problem in standard form:

\[\begin{align*}
\text{Max} & \quad c^T x \\
\text{st:} & \quad Ax \leq b \\
& \quad x \geq 0 \\
A & \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c \in \mathbb{R}^n, \quad x \in \mathbb{R}^n
\end{align*}\]

\[\begin{align*}
\zeta &= \sum_{j=1}^{m} \gamma_j x_j \\
\beta_i - \sum_{j=1}^{n} \alpha_j &= \ldots \\
\beta_i \geq 0 &= \ldots
\end{align*}\]

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1. In dictionaries, except \(\zeta\), the variables that appear on the left are called basic variables, while those on the right are called nonbasic variables, which are set to zero when finding feasible solutions. B denotes the basic variable set whereas N denotes the nonbasic variable set.

2. Whether a dictionary is feasible or not is decided by \(\beta_i\) coefficients of this dictionary. If and only if all \(\beta_i\) are nonnegative, the dictionary is feasible. Otherwise, it is infeasible. When the dictionary is feasible, we can set the nonbasic variables to zero so that we can easily get a feasible solution to the original problem. With an infeasible dictionary, however, it is not always easy to find a feasible solution. So we need to solve an auxiliary problem to get one.
**Phase 1  Auxiliary problem**
This step will be discussed in the section “Auxiliary Problem”

**Phase 2  Optimal solution**

**Step 2** Choose an incoming variable $x_k$ from nonbasic variable set N
Pick k from set $\{ j \in N : \gamma_j > 0 \}$

\[
\begin{align*}
\left\{ i \in B : \begin{array}{c}
\frac{\tilde{a}}{\alpha} \\
\tilde{\alpha}
\end{array} \right\}
\end{align*}
\]

\[
\begin{align*}
\xi & \leq \sum_{j=1}^{n} \gamma_j, \\
\bar{\beta} & - \sum \bar{\alpha} \quad , \quad i \in B
\end{align*}
\]

\[
\begin{align*}
x & \leq \\
- & \leq \\
- & \leq \\
\geq
\end{align*}
\]

---

3 We put bars over the coefficients to indicate that they change as the algorithm progresses.
4 There are several selection criteria beside the one used here.
The objective function is to maximize $\zeta$. Choose $x_1$ as the incoming variable (step 2). When increasing $x_1$ with a small number $\varepsilon$ and keeping other nonbasic variables $x_2$ and $x_3$ unchanged, the value of the objective function is increased. At the same time, however, basic variables will fall to constraints ($\geq 0$). In order to keep these constraints $x_1$ can only increase to a certain value. This value is decided by minimum of $\{3/1, 4/2, 2/2\}$. The corresponding outgoing variable is $x_7$ (step 3). Then a new dictionary is constructed.

Through the row operation related to $x_1$ and $x_7$, we have:

$$
x_1 = 1 - \frac{3}{2} x_2 - \frac{1}{2} x_3 - \frac{1}{2} x_7
$$

$$
x_4 = 2 - \frac{3}{2} x_2 - \frac{3}{2} x_3 - \frac{1}{2} x_7
$$

$$
x_5 = 3 - \frac{3}{2} x_2 - \frac{5}{2} x_3 - \frac{1}{2} x_7
$$

$$
x_6 = 2 - 4 x_2 - 3 x_3 - x_7
$$

$$
\zeta = 5 - \frac{5}{2} x_2 - \frac{11}{2} x_3 - \frac{5}{2} x_7
$$

$$
\arg\left\{\min\left\{\frac{\varepsilon}{\alpha_3}, \frac{\zeta}{\alpha_3}, 0\right\}\right\} = \arg\left\{\min\left\{\frac{1}{3}, \frac{6}{5}, \frac{2}{3}\right\}\right\} = 6
$$
\[
x_3 = \frac{2}{3} x_2 - \frac{4}{3} x_2 - \frac{1}{3} x_6 - \frac{1}{3} x_7
\]
\[
x_1 = \frac{4}{3} x_2 - \frac{5}{6} x_2 - \frac{1}{6} x_6 - \frac{1}{3} x_7
\]
\[
x_4 = 1 - \frac{7}{2} x_2 - \frac{1}{2} x_6
\]
\[
x_5 = \frac{4}{3} - \frac{29}{6} x_2 + \frac{5}{6} x_6 - \frac{4}{3} x_7
\]
\[
\zeta = \frac{26}{3} - \frac{29}{6} x_2 + \frac{11}{6} x_6 - \frac{2}{3} x_7
\]

\[
\arg\left\{ \min \left\{ -\frac{1}{\alpha_2} f(x), \bar{\alpha}_2 \right\} \right\} = \arg\left\{ \min \left\{ \frac{8}{5}, \frac{2}{7}, \frac{8}{29} \right\} \right\} = 6
\]
\[
x_2 = \frac{1}{29} (8 - 8x_7 - 5x_6 - 6x_3)
\]
\[
x_3 = \frac{1}{29} (30 - x_7 - 3x_6 - 8x_3)
\]
\[
x_1 = \frac{1}{29} (32 - 3x_7 - 9x_6 - 5x_3)
\]
\[
x_4 = \frac{1}{29} (1 - 28x_7 - 3x_6 - 21x_3)
\]
\[
\zeta = 10 - x_5 - x_6 - 2x_7
\]

Due to all negative, the progress stops. We get an optimal solution:

\[
\zeta = 10
\]
\[
(x_1, x_2, x_3) = (\frac{32}{29}, \frac{8}{29}, \frac{30}{29})
\]

\[
\beta_i \geq
\]
\[\begin{align*}
3x_1 - 2x_2 & \leq 5 \\
1 - 2x_3 & \leq -1 \\
2 - 5x_2 - x_3 & \leq -2 \\
x_1 - 2x_2 + x_3 & \geq 0
\end{align*}\]

\[\beta_i \geq 0\]

\[x = x\]

\[\begin{array}{c}
-x \\
- & \leq \\
- & \leq \\
- & \leq \\
\geq
\end{array}\]

\[\zeta = -
\]

\[\begin{array}{c}
-x \\
- & \leq \\
- & \leq
\end{array}\]

\[\zeta = -
\]

\[\begin{align*}
x_2 & = \frac{1}{6}x_1 + \frac{1}{6}x_2 - \frac{1}{6}x_3 \\
x_0 & = \frac{7}{6} - \frac{1}{2}x_1 - x_3 + \frac{1}{6}x_6 - \frac{5}{6}x_5 \\
x_4 & = 6 - 2x_1 - 2x_3 - x_5 \\
\zeta & = \frac{7}{6} - \frac{1}{2}x_1 - x_3 - \frac{1}{6}x_6 - \frac{5}{6}x_5
\end{align*}\]
This dictionary is optimal for the auxiliary problem. We now drop $x_0$ from the equations and replace the original objective function with $x \ 2 \ 3$ and $x \ 3$. The feasible dictionary of the original problem is:

\[
x_3 : \frac{7}{6} - \frac{1}{2} x_1 - x_0 \ \frac{1}{6} x_6 - \frac{5}{6} x_3 \\
x_2 : \frac{1}{6} - \frac{1}{2} x_1 - \frac{1}{6} x_6 - \frac{1}{6} x_3 \\
x_4 : \frac{3}{11} - x_1 - 2 x_0 - \frac{1}{3} x_6 - \frac{2}{3} x_3 \\
\zeta = 0 - x_0
\]

\[
x_3 : \frac{7}{6} - \frac{1}{2} x_1 - \frac{1}{6} x_6 - \frac{5}{6} x_3 \\
x_2 : \frac{1}{6} - \frac{1}{2} x_1 - \frac{1}{6} x_6 - \frac{1}{6} x_3 \\
x_4 : \frac{3}{11} - x_1 - \frac{1}{3} x_6 - \frac{2}{3} x_3 \\
\zeta = \frac{5}{6} - \frac{3}{2} x_1 - \frac{1}{6} x_6 + \frac{7}{6} x_3
\]

\[
= - - - \\
\zeta = - - - \quad \{ i \in B: \quad \Gamma \alpha - \Gamma \bar{\alpha} \}
\]

\[
\forall \in N \ \alpha \leq \gamma \quad \quad \gamma_i \leq
\]

\[
= - - - \\
= - - - \\
\zeta - -
\]
This time, \( x_3 = x_3 \) and to \( x_4, x_5, x_6 \) respectively \( x_3 \) fall to the same constrain \((\leq 1/2)\). If we choose outgoing variable \( x_j = x_4 \), we get a new dictionary:

We can keep constructing new dictionaries, but the objective value is never increased and finally we meet the dictionary we got in advance. Then, a cycle appears.

This section will be discussed further at next lecture.