For any hypergraph \( \mathcal{H} \subseteq 2^V \) we have

\[
\mathcal{H}^+ \cap \mathcal{H}^* = \emptyset \quad \text{and} \quad \mathcal{H}^+ \cup \mathcal{H}^* = 2^V.
\]
For any hypergraph $\mathcal{H} \subseteq 2^V$ we have

$$\mathcal{H}^+ \cap \mathcal{H}^{*-} = \emptyset \quad \text{and} \quad \mathcal{H}^+ \cup \mathcal{H}^{*-} = 2^V.$$  

Given a Sperner hypergraph $\mathcal{H}$, we can view it as the family of minimal sets in the monotone system $\mathcal{H}^+$. Similarly, $\mathcal{H}^*$ can be viewed as the family of maximal independent sets in the independence system $\mathcal{H}^{*-}$. 

**Notations**
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We shall assume that $\mathcal{H}$ or equivalently $\mathcal{H}^*$ are represented by a membership oracle $\Omega$, either for the monotone system $\mathcal{H}^+$ or for the independence system $\mathcal{H}^*$ (one is simply the negation of the other).
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We shall assume that $\mathcal{H}$ or equivalently $\mathcal{H}^*$ are represented by a membership oracle $\Omega$, either for the monotone system $\mathcal{H}^+$ or for the independence system $\mathcal{H}^{*-}$ (one is simply the negation of the other).

The set $B(\Omega) = \mathcal{H} \cup \mathcal{H}^*$ is called the boundary of $\Omega$ (as well as of the independence system $\mathcal{H}^{*-}$ and the monotone system $\mathcal{H}^+$).
Fact 1. Consider an arbitrary algorithm $A$, which generates every Sperner family $\mathcal{H}$ by using only a membership oracle $\Omega$ for the monotone system $\mathcal{H}^+$. Then, for every hypergraph $\mathcal{H}$ such an algorithm must call the oracle at least $|B(\Omega)| = |\mathcal{H}| + |\mathcal{H}^*|$ times.
**Easy Facts**

**Fact 1.** Consider an arbitrary algorithm \( A \), which generates every Sperner family \( \mathcal{H} \) by using only a membership oracle \( \Omega \) for the monotone system \( \mathcal{H}^+ \). Then, for every hypergraph \( \mathcal{H} \) such an algorithm must call the oracle at least \( |\mathcal{B}(\Omega)| = |\mathcal{H}| + |\mathcal{H}^*| \) times.

**Proof.** For any \( B \in \mathcal{B}(\Omega) \) let \( \Omega_B \) be defined by

\[
\Omega_B(S) = \begin{cases} 
\Omega(S) & \text{if } S \neq B, \\
\overline{\Omega}(S) & \text{if } S = B.
\end{cases}
\]

Then, \( \Omega_B \) is the membership oracle of a monotone system corresponding to a different Sperner hypergraph, and thus algorithm \( A \) must distinguish the systems represented by \( \Omega \) and \( \Omega_B \), for any \( B \in \mathcal{B}(\Omega) \). \( \square \)
Corollary 1. A purely membership oracle based generation of $\mathcal{H}$ (or $\mathcal{H}^*$) will (anyway) generate both sets $\mathcal{H} \cup \mathcal{H}^*$. 
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Corollary 2. Generating the maximal independent sets \( \text{max} \mathcal{I} \) of an independence system \( \mathcal{I} \) represented by a membership oracle \( \Omega \) is not simply \text{NP-hard} (cf. Lawler, Lenstra and Rinnooy Kan, 1980) but \text{exponential}, in worst case!!! Because, for infinite families of independence systems we have

\[
|\min \left( 2^V \setminus \mathcal{I} \right)| \gg \text{poly}(|\text{max} \mathcal{I}|, |\Omega|)
\]

for any polynomial \( \text{poly}() \).
Corollary 3. A purely membership oracle based generation of $\mathcal{H}$ can be efficient in total time only if

$$|\mathcal{H}^*| \leq poly(|\mathcal{H}|, |\Omega|)$$

for some polynomial $poly()$.

If such an inequality hold, we say that $\mathcal{H}$ is dual bounded.
Corollary 3. A purely membership oracle based generation of $\mathcal{H}$ can be efficient in **total time** only if

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---

**We do not know** if there is a purely membership oracle based algorithm $A$ which would generate in **polynomial total time** every dual bounded hypergraph.

**BUT**

**We do know** that there is one, which does that in **quasi-polynomial total time**.
The Idea of Joint Generation
(Bioch and Ibaraki, 1995, Gurvich and Khachiyan, 1998)

Given the Sperner hypergraphs $S \subseteq \mathcal{H}$ and $Q \subseteq \mathcal{H}^*$, we have

\[
S = \mathcal{H} \text{ and } Q = \mathcal{H}^*
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\[
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The latter conditions do not depend on $\mathcal{H}$ or $\mathcal{H}^*$!!
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Given the Sperner hypergraphs $S \subseteq \mathcal{H}$ and $Q \subseteq \mathcal{H}^*$, we have

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$$\iff$$

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$$\iff$$

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**Dualization**: Given the Sperner hypergraphs $S \subseteq \mathcal{H}$ and $Q \subseteq \mathcal{H}^*$, the equality $S^+ \cup Q^- = 2^V$ can be tested, and if it does not hold, a new subset $X \in 2^V \setminus (S^+ \cup Q^-)$ can be found in $O(N^{o(\log N)} \log N)$ time, where $N = |S| + |Q|$. (Fredman and Khachiyan, 1996)
Joint Generation
(Bioch and Ibaraki, 1995, Gurvich and Khachiyan, 1998)

Given an oracle $\Omega$ representing the monotone system $\mathcal{H}^+ \subseteq 2^V$ (or the independence system $\mathcal{H}^{*-} \subseteq 2^V$), let us start with $S = Q = \emptyset$.

While $S^+ \cup Q^- \neq 2^V$ do:
  
  Let $X \in 2^V \setminus (S^+ \cup Q^-)$ be the set found by dualization.
  Test $\Omega(X) = ?$ (remember, we have $\mathcal{H}^+ \cup \mathcal{H}^{*-} = 2^V$).
  If $X \in \mathcal{H}^+$, then by calling the oracle at most $|X|$-times,
    find a set $H \in \mathcal{H}$ for which $X \supseteq H$, and
    set $S = S \cup \{H\}$.
  If $X \in \mathcal{H}^{*-}$, then by calling the oracle at most $|V \setminus X|$-times
    find a set $H \in \mathcal{H}^*$ for which $X \subseteq H$, and
    set $Q = Q \cup \{H\}$.

The complexity of one incremental step is $O(N^o(\log N) + O(n|\Omega|)$ time, where $N = |S| + |Q|$ and $n = |V|$. 
Theorem. Given an oracle \( \Omega \) representing a monotone system \( \mathcal{H}^+ \), joint generation generates the boundary family \( B(\Omega) = \mathcal{H} \cup \mathcal{H}^* \) in incremental quasi-polynomial time.
**Theorem.** Given an oracle $\Omega$ representing a monotone system $H^+$, joint generation generates the boundary family $B(\Omega) = H \cup H^*$ in incremental quasi-polynomial time.

**Consequence.** If $H$ is dual bounded, i.e. $|H^*| \leq \text{poly}(|H|,|\Omega|)$, then joint generation generates $H$ in quasi-polynomial total time (just do not output sets in $H^*$).
Joint Generation
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**Theorem.** Given an oracle \( \Omega \) representing a monotone system \( \mathcal{H}^+ \), joint generation generates the boundary family \( \mathcal{B}(\Omega) = \mathcal{H} \cup \mathcal{H}^* \) in incremental quasi-polynomial time.

**Consequence.** If \( \mathcal{H} \) is dual bounded, i.e. \( |\mathcal{H}^*| \leq poly(|\mathcal{H}|, |\Omega|) \), then joint generation generates \( \mathcal{H} \) in quasi-polynomial **total** time (just do not output sets in \( \mathcal{H}^* \)).

**Remark.** Joint generation may not generate \( \mathcal{H} \) incrementally efficiently. For instance, we may get all the sets of \( \mathcal{H}^* \) before getting any from \( \mathcal{H} \)!
**Fact 2.** If $S \subseteq \mathcal{H}$, $Q \subseteq \mathcal{H}^*$, and $Q \not\subseteq S^*$, then for any $Q \in Q \setminus S^*$ we have a $j \in V \setminus Q$ such that $X = Q \cup \{j\} \not\in S^+$. 

For such a set we **must have** $X \in \mathcal{H}^+ \setminus S^+$. 
Fact 2. If $S \subseteq \mathcal{H}$, $Q \subseteq \mathcal{H}^*$, and $Q \not\subseteq S^*$, then for any $Q \in Q \setminus S^*$ we have a $j \in V \setminus Q$ such that $X = Q \cup \{j\} \not\in S^+$. 

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Modifying Joint Generation

**Fact 2.** If $S \subseteq \mathcal{H}$, $Q \subseteq \mathcal{H}^*$, and $Q \nsubseteq S^*$, then for any $Q \in Q \setminus S^*$ we have a $j \in V \setminus Q$ such that $X = Q \cup \{j\} \notin S^+$.

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Fact 2. If $S \subseteq H$, $Q \subseteq H^*$, and $Q \not\subseteq S^*$, then for any $Q \in Q \setminus S^*$ we have a $j \in V \setminus Q$ such that $X = Q \cup \{j\} \not\subseteq S^*$.

For such a set we **must have** $X \in H^+ \setminus S^*$. 
Fact 2. If $S \subseteq \mathcal{H}$, $Q \subseteq \mathcal{H}^*$, and $Q \not\subseteq S^*$, then for any $Q \in Q \setminus S^*$ we have a $j \in V \setminus Q$ such that $X = Q \cup \{j\} \not\subseteq S^*$.

For such a set we must have $X \in \mathcal{H}^+ \setminus S^+$.

The condition $Q \not\subseteq S^*$ can be tested, and if it holds, then a set $X \in \mathcal{H}^+ \setminus S^+$ can be constructed in $O(n|S||Q|)$ time. We do not need the oracle $\Omega$ for this.
Modified Joint Generation

While \( Q \not\subseteq S^* \),

let \( X \in \mathcal{H}^+ \setminus S^+ \) be constructed as above, and
by calling the oracle at most \( |X| \)-times,
find a set \( H \in \mathcal{H} \) for which \( X \supseteq H \), and
set \( S = S \cup \{H\} \).

If \( Q \subseteq S^* \) and \( S^+ \cup Q^- \neq 2^V \)
then let \( X \in 2^V \setminus (S^+ \cup Q^-) \) be found by dualization.

Test \( \Omega(X) = ? \) (remember, we have \( \mathcal{H}^+ \cup \mathcal{H}^{*-} = 2^V \)).

If \( X \in \mathcal{H}^+ \), then by calling the oracle at most \( |X| \)-times,
find a set \( H \in \mathcal{H} \) for which \( X \supseteq H \), and
set \( S = S \cup \{H\} \).

If \( X \in \mathcal{H}^{*-} \), then by calling the oracle at most \( |V \setminus X| \)-times
find a set \( H \in \mathcal{H}^* \) for which \( X \subseteq H \), and
set \( Q = Q \cup \{H\} \).

Return to the while loop.
Fact 2. In every iteration of the modified joint generation we have

$$|Q| \leq |S^* \cap H^*| + 1.$$ 

Corollary 4. Joint generation becomes incrementally efficient (quasi-polynomial) if for all subfamilies $S \subseteq H$ the following inequality holds:

$$|S^* \cap H^*| \leq poly(|\Omega|, |S|).$$

Let us call a hypergraph $H$ (represented by a membership oracle $\Omega$) uniformly dual bounded if the above inequality holds for all subfamilies $S \subseteq H$. 
Uniformly Dual Bounded Hypergraphs

Fact 2. In every iteration of the modified joint generation we have

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Let us call a hypergraph \( H \) (represented by a membership oracle \( \Omega \)) **uniformly dual bounded** if the above inequality holds for all subfamilies \( S \subseteq H \).

Corollary 5. Since dualization is used as a black-box, joint generation may increment \( Q \) consecutively, up to \( S^* \cap H^* \) before incrementing \( S \). Thus, joint generation is incrementally efficient to generate \( H \) if and only if \( H \) is uniformly dual bounded.