Monotone Systems of Linear Inequalities

Let \( m, n \in \mathbb{Z}_+ \), \( A \in \mathbb{R}^{m \times n}_+ \), \( b \in \mathbb{R}^m \) and consider the following system

\[
Ax \geq b, \quad x \in \mathbb{Z}^n
\]

- Generate all \textbf{minimal feasible solutions}

  integer programming ... (Lawler, Lenstra and Rinnooy Kan, 1980)

- Generate all \textbf{maximal infeasible solutions}

  resource constrained scheduling ... (Stork and Uetz, 2002)
Monotone Systems of Linear Inequalities

- **Oracle**: \( m, n \in \mathbb{Z}_+ \), a matrix \( A = [a_1, \cdots, a_n] \in \mathbb{R}^{m \times n}_+ \), and a vector \( b \in \mathbb{R}^m \), and define

\[
f(S) \overset{\text{def}}{=} \sum_{j \in S} a_j.
\]

- **Target pair of hypergraphs**:

\[
\mathcal{H} = \{ \text{minimal subsets } F \text{ for which } f(F) \geq b \}
\]

\[
\mathcal{H}^* = \{ \text{maximal subsets } I \text{ for which } f(I) \not\geq b \}
\]
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- **Oracle**: $m, n \in \mathbb{Z}_+$, a matrix $A = [a_1, \ldots, a_n] \in \mathbb{R}_+^{m \times n}$, and a vector $b \in \mathbb{R}^m$, and define

  \[ f(S) \overset{\text{def}}{=} \sum_{j \in S} a_j. \]

- **Target pair of hypergraphs**:
  \[
  \mathcal{H} = \{ \text{minimal subsets } F \text{ for which } f(F) \geq b \}
  \]
  \[
  \mathcal{H}^* = \{ \text{maximal subsets } I \text{ for which } f(I) \not\geq b \}
  \]

- **Then**, $\mathcal{H}$ is uniformly dual bounded, while $\mathcal{H}^*$ is not. More precisely,

  for any subfamily $S \subseteq \mathcal{H}$ we have \(|S^* \cap \mathcal{H}^*| \leq mn|S|\),

  while for infinitely many $m, n$ there exists $A$ and $b$ such that

  for some subfamily $S^* \subseteq \mathcal{H}^*$ we have \(|S \cap \mathcal{H}| > \exp(m, n, |S^*|)\).
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• **Oracle**: \( m, n \in \mathbb{Z}_+ \), a matrix \( A = [a_1, \cdots, a_n] \in \mathbb{R}_+^{m \times n} \), and a vector \( b \in \mathbb{R}^m \), and define

\[
    f(S) \overset{\text{def}}{=} \sum_{j \in S} a_j.
\]

• **Target pair of hypergraphs**:

\( \mathcal{H} = \{ \text{minimal subsets } F \text{ for which } f(F) \geq b \} \)

\( \mathcal{H}^* = \{ \text{maximal subsets } I \text{ for which } f(I) \nleq b \} \)

• \( \mathcal{H} \) can be generated in incremental quasi-polynomial time, while the generation of \( \mathcal{H}^* \) is NP-hard!

(c.f. Lawler, Lenstra, Rinnooy Kan, 1980)