Minimal Universal Bipartite Graphs

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Abstract

A graph $U$ is (induced)-universal for a class of graphs $X$ if every member of $X$ is contained in $U$ as an induced subgraph. We study the problem of finding a universal graph with minimum number of vertices for various classes of bipartite graphs: exponential classes of bipartite (and general) graphs, bipartite chain graphs, bipartite permutation graphs, and general bipartite graphs. For exponential classes and general bipartite graphs we present a construction which is asymptotically optimal while for the other classes our solutions are optimal in order.

1 Introduction

Denote by $\Gamma_n$ the class of all simple (undirected, without loops and multiple edges) graphs with vertex set $\{1, 2, \ldots, n\}$ and let $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. Given a graph $G \in \Gamma$, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. Also, $|G| = |V(G)|$ is the order of $G$. If $W$ is a subset of vertices of $G$, then $G[W]$ is the subgraph of $G$ induced by $W$, i.e. the subgraph of $G$ with the vertex set $W$ and two vertices being adjacent if and only if they are adjacent in $G$. If a graph $H$ is isomorphic to an induced subgraph of $G$, we say that $H$ is embeddable into $G$.

Throughout this paper we use the following notation: $\langle n \rangle = \{1, 2, \ldots, n\}$.

By $\mathcal{E}_{i,j}$ we denote the class of graphs whose vertices can be partitioned into at most $i$ independent sets and $j$ cliques. In particular, $\mathcal{E}_{2,0}$ is the class of bipartite graphs, $\mathcal{E}_{0,2}$ is the class of co-bipartite graphs, and $\mathcal{E}_{1,1}$ is the class of split graphs [12].

A class of graphs $X \subseteq \Gamma$ is called hereditary if $G \in X$ implies $H \in X$ for every graph $H$ isomorphic to an induced subgraph of $G$. Let us denote $X_n = X \cap \Gamma_n$.

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For an arbitrary hereditary class $X$, a graph $UX_n$ is called an $n$-universal $X$-graph if every graph in $X_n$ is isomorphic to an induced subgraph of $UX_n$. From obvious cardinality arguments, we have

$$\log_2 |X_n| \leq n \log_2 |UX_n|.$$ 

Also, we trivially have

$$n \log_2 n \leq n \log_2 |UX_n|.$$ 

A sequence of universal $X$-graphs $\{UX_n, n = 1, 2, \ldots\}$ will be called asymptotically optimal if

$$\lim_{n \to \infty} \frac{n \log_2 |UX_n|}{\max \left( \log_2 |X_n|, n \log_2 n \right)} = 1.$$ 

and optimal in order (order-optimal) if there is a constant $c$ such that for any $n \geq 1$,

$$\frac{n \log_2 |UX_n|}{\max \left( \log_2 |X_n|, n \log_2 n \right)} \leq c.$$ 

In the present paper we construct optimal universal graphs for several families of bipartite graphs, such as general bipartite graphs, bipartite permutation graphs as well as some specific families defined in the next section.

2 Preliminaries

It has been proven in [2] that for any infinite hereditary class $X$ different from the class of all graphs,

$$\lim_{n \to \infty} \frac{\log_2 |X_n|}{\binom{n}{2}} = 1 - \frac{1}{k(X)},$$ 

where $k(X)$ is a natural number called index of the class $X$ (similar results can also be found in [7]). The index $k(X)$ of a class $X$ is the maximum $k$ such that $X$ contains a class $E_{i,j}$ with $i + j = k$. Let us extend this definition by assuming that the index of any finite hereditary class is 0, and the index of the class of all graphs is infinity. With this extension, the family of all hereditary classes is partitioned into countable number of strata, each of which consists of classes with the same index. Moreover, the classes $E_{i,j}$ with the same value of $i + j$ are the only minimal classes in the respective stratum. In particular, for $k = 2$ there are exactly three minimal classes: bipartite graphs, complements of bipartite graphs, and split graphs. Therefore, an infinite hereditary class of graphs has index 1 if and only if it contains none of the three listed classes. The classes of index 1 and the respective stratum has been called in [3] unitary. The unitary stratum is of particular interest for several reasons. First, the universal algorithm proposed in [1] for asymptotically
optimal representation of graphs in any non-unitary class $X$ does not work for unitary classes, since the equality (1) does not provide the asymptotic behavior of $\log_2|X_n|$ when $k(X) = 1$. Second, the unitary stratum contains many classes of theoretical and practical importance, such as forests, planar, interval, permutation, chordal bipartite, line, threshold [16] graphs, cographs, etc. In order to provide a differentiation of the unitary classes in accordance with their size, let us introduce the following definition: two graph classes $X$ and $Y$ will be called isometric if there are positive constants $c_1$, $c_2$ and $n_0$ such that $|Y_n|^{c_1} \leq |X_n| \leq |Y_n|^{c_2}$ for any $n > n_0$.

Clearly this isometry is an equivalence relation. The equivalence classes of this relation will be called layers.

All finite classes of graphs constitute a single layer, and all classes of index greater than 1 also constitute a single layer. Between these two extremes lies the unitary stratum, and it consists of infinitely many layers. To see this, consider the class $Z^p$ of bipartite graphs containing no $K_{p,p}$ as an induced subgraph. From the well-known results on the maximum number of edges in graphs in $Z^p$ (see e.g. [6, 10]), we have:

$$c_1 n^{2 - \frac{2}{p+1}} < \log_2 |Z^p_n| < c_2 n^{2 - \frac{1}{p}} \log_2 n.$$  

This implies, in particular, that $Z^p$ and $Z^{2p}$ are non-isometric.

The first four lower layers in the unitary stratum have been distinguished in [17]:

- **constant** layer contains classes $X$ with $\log_2 |X_n| = O(1)$,
- **polynomial** layer contains classes $X$ with $\log_2 |X_n| = O(\log_2 n)$,
- **exponential** layer contains classes $X$ with $\log_2 |X_n| = O(n)$,
- **factorial** layer contains classes $X$ with $\log_2 |X_n| = O(n \log_2 n)$.

Independently, the same result has been obtained by Alekseev in [3]. Moreover, Alekseev provided the first four layers with the description of all minimal classes, which has led to the complete structural characterization of the first three layers (some more involved results can be found in [5]). In particular, the structure of exponential classes of graphs can be characterized as follows.

**Theorem 1** For each exponential class $X$, there is a constant $p$ such that every graph $G \in X$ can be partitioned into at most $p$ subsets each of which is either an independent set or a clique and between any two subsets there are either all possible edges or none of them.

This characterization shows that all exponential classes have a rather simple structure, which leads in particular to a simple construction of order-optimal universal graphs for classes in this layer (Section 3).
The factorial layer is substantially richer. In fact, most of the unitary classes mentioned above are factorial (the unique exception in the above list is the class of chordal bipartite graphs, which is superfactorial [18]) and most of the works on induced-universal graphs relate to factorial classes, such as trees (forests), planar graphs, or graphs of bounded arboricity [4, 8, 15]. In the present paper we supplement this list with two new results: asymptotically optimal universal graphs for minimal factorial classes of bipartite graphs (Section 4) and order-optimal bipartite permutation graphs (Section 5).

According to the results in [3], the factorial layer has 9 minimal classes, three of which are subclasses of bipartite graphs, another three are subclasses of co-bipartite graphs and the remaining three are subclasses of split graphs. The three minimal factorial classes of bipartite graphs are:

\( \mathcal{P}_1 \) the class of 2\( K_2 \)-free bipartite graphs, also known as chain graphs [19], difference graphs [14] or bisplit graphs [13];

\( \mathcal{P}_2 \) the class of graphs with vertex degree at most 1;

\( \mathcal{P}_3 \) the class of bipartite complements of graphs in \( \mathcal{P}_2 \), also known as almost complete bipartite graphs.

Very little is known about universal graphs for non-unitary classes. In Section 6) we describe asymptotically optimal universal graphs for the class of general bipartite graphs, which is one of the three minimal non-unitary classes.

3 Exponential classes of graphs

Let \( X \) be an exponential class of graphs and \( k \) a constant associated to it. The \( n \)-universal \( X \)-graph \( UX_n \) is defined as follows: Let \( \bar{\Gamma}_k \) contain exactly one graph from each isomorphism class of \( \Gamma_k \).

(a) The vertex set of \( UX_n \) is \( V(UX_n) = \{ (\bar{G}, i, j, \delta) \mid \bar{G} \in \bar{\Gamma}_k, i \in \langle k \rangle, j \in \langle n \rangle, \delta \in \{0, 1\} \} \).

(b) Two distinct vertices \( (\bar{G}_1, i_1, j_1, \delta_1) \) and \( (\bar{G}_2, i_2, j_2, \delta_2) \) are adjacent in \( UX_n \) if and only if \( \bar{G}_1 = \bar{G}_2 \) and either \( i_1 i_2 \in E(\bar{G}_1) \) or \( i_1 = i_2, \delta_1 = \delta_2 = 1 \).

First let us show that the constructed graph is indeed \( n \)-universal for the class \( X \).

Theorem 2 Every \( n \)-vertex graph in \( X \) is embeddable into \( UX_n \).

Proof. Let \( G \) be a graph with \( n \) vertices in \( X \). Since \( X \) is an exponential class, the vertices of \( G \) can be partitioned into independent sets \( V_1, \ldots, V_r \) and cliques \( V_{r+1}, \ldots, V_p \) with \( p \leq k \) such that if two vertices \( u \) and \( v \) belong to the
same subset $V_i$ then $N_{G\setminus V_i}(u) = N_{G\setminus V_i}(v)$. For each subset let us define a bijection

$$\phi_i : V_i \rightarrow \langle |V_i| \rangle \subset \langle n \rangle.$$  

By contracting each subset $V_j$ into a single vertex $v_j$ we obtain a new graph $H$ with at most $k$ vertices. Then $H$ is isomorphic to an induced subgraph of some $\tilde{H} \in \overline{\Gamma}_k$; let the isomorphism be given by $\psi : V(H) \rightarrow V(\tilde{H}) = \langle k \rangle$. It is easily verified that mapping a vertex $v \in V_i$ to $(\tilde{H}, \psi(v_i), \phi_i(v), \delta_i)$, where

$$\delta_i = \begin{cases} 
0 & \text{if } i \leq r \\
1 & \text{if } i > r 
\end{cases}$$ 

provides us with an embedding of $G$ into $U X_n$. ■

Since $\log_2 |X_n| = O(n)$ for any exponential class $X$ we now conclude that

**Theorem 3** The graph $U X_n$ defined by (a) and (b) is asymptotically optimal for the class $X$.

## 4 Minimal factorial classes of bipartite graphs

In this section we show that for each of the three minimal factorial classes of bipartite graphs $\mathcal{P}_j$ ($j = 1, 2, 3$) there is an $n$-universal $\mathcal{P}_j$-graph with $2n$ vertices. For $j = 2$ and $j = 3$, the statement is trivial. Now we prove it for $j = 1$, i.e. for the class of chain graphs. To this end, let us introduce the following definitions and notations.

A bipartite graph will be called *prime* if it is connected and any two distinct vertices of the graph have different neighborhoods. It is known (see e.g. [11]) that in a prime chain graph $G$ with parts $V_1$ and $V_2$, the cardinality of $V_1$ equals the cardinality of $V_2$.

Denote by $H_{n,m}$ the graph with $nm$ vertices which can be partitioned into $m$ independent sets $V_1 = \{v_{1,1}, \ldots, v_{1,n}\}, \ldots, V_m = \{v_{m,1}, \ldots, v_{m,n}\}$ so that for each $i = 1, \ldots, n - 1$ and for each $j = 1, \ldots, n$, vertex $v_{i,j}$ is adjacent to vertices $v_{i+1,1}, v_{i+1,2}, \ldots, v_{i+1,n}$ and there are no other edges in the graph. In other words, every two consecutive independent set induce in $H_{n,m}$ a prime chain graph. The graph $H_{n,m}$ will be called *canonical*. An example of a canonical graph is given in Figure 1.

**Theorem 4** The graph $H_{2,n}$ is an $n$-universal chain graph.

**Proof.** Let $G$ be an $n$-vertex chain graph with parts $V_1$ and $V_2$. We shall create a graph $H_{2,n}$ containing $G$ by adding to $G$ some new vertices and edges. To this end, we partition $V_1$ and $V_2$ into modules (i.e. subset of vertices with the same neighborhood) and denote the modules of $V_1$ by $V_1^1, V_1^3, V_1^5, \ldots, V_1^{2p-1}$, and the modules of $V_2$ by $V_2^2, V_2^4, V_2^6, \ldots, V_2^{2p}$ (observe that the number of modules in $V_1$ and $V_2$ must be equal). Now, for each odd $i = 1, \ldots, 2p - 1$, we create a set of new vertices $V_i$ of size $|V_i|$, and for each even $j = 2, \ldots, 2p$, we create a set of new vertices $V_j$ of size $|V_j|$. The desired graph $H_{2,n}$ will contain two parts of vertices $V'_1 = V_1^1 \cup V_1^3 \cup \ldots \cup V_1^{2k-1} \cup V_1^{2k}$ and $V'_2 = V_2^2 \cup V_2^4 \cup \ldots \cup V_2^{2k-1} \cup V_2^{2k}$ of the
same size. To complete the construction, we first re-index the vertices in $V'_1$ and $V'_2$ consecutively, following the order of subsets, and then for each $j = 1, 2, \ldots, |V'_2|$, we connect by an edge the $j$-th vertex of $V'_2$ to the (not yet adjacent) $i$-th vertex of $V'_1$ for each $i = 1, 2, \ldots, j$. According to the construction, the obtained graph $H_{2,m}$ is clearly a prime chain graph. Moreover, it contains $G$ as an induced subgraph since no new edge connects two old vertices. ■

From definition of factorial classes and Theorem 4 we conclude that

**Corollary 1** Graph $H_{2,n}$ is an order-optimal universal chain graph.

### 5 Bipartite permutation graphs

In this section we extend the result of the previous one to the class of bipartite permutation graphs. We also use the notation introduced in the previous section. The result presented here is based on Theorem 5 proven in [9], where a vertex ordering "<" is called *increasing* if $x < y$ implies $N(x) \subseteq N(y)$, and *decreasing* if $x < y$ implies $N(y) \subseteq N(x)$.

**Theorem 5** A connected graph $G$ is bipartite permutation if and only if the vertex set of $G$ can be partitioned into independent sets $D_0, D_1, \ldots, D_q$ so that

(a) any two vertices in non-consecutive sets are non-adjacent,

(b) any two consecutive sets $D_{j-1}$ and $D_j$ induce a chain graph, denoted $G_j$, 

Figure 1: Canonical graph $H_{5,5}$
(c) for each $j = 1, 2, \ldots, q - 1$, there is an ordering of vertices in set $D_j$, which is decreasing for $G_j$ and increasing for $G_{j+1}$.

With every connected bipartite permutation graph $G$ we shall associate a partition as in Theorem 5 and the respective independent sets $D_0, D_1, \ldots, D_q$ will be called the layers of $G$. Now let us use the above theorem in order to prove the following result.

**Theorem 6** The graph $H_{n,n}$ is an $n$-universal bipartite permutation graph.

**Proof.** Let us first notice that without loss of generality we may restrict ourselves to connected bipartite permutation graphs $G$, since the "principal" subgraphs of $H_{n,n}$ (i.e. those located on the "main diagonal" of $H_{n,n}$) are universal for connected components of $G$.

The proof will be given by induction on the number of layers in $G$. The basis of the induction is established in Theorem 4. Now assume that the theorem is valid for any bipartite permutation graph with $k \geq 2$ layers, and let $G$ be an $n$-vertex graph with $k + 1 \leq n$ layers. For $j = 1, \ldots, k + 1$, let $V_j$ denote the set of vertices in the $j$-th layer of $G$, $n_j = |V_j|$ and also $m = n_1 + \ldots + n_k$.

Let $H_{k,m}$ be a canonical graph containing the first $k$ layers of $G$ as an induced subgraph. Now we create an auxiliary graph $H'$ out of $H_{k,m}$ by

1. adding to $H_{k,m}$ the set of vertices $V_{k+1}$,
2. connecting the vertices of $V_k$ (belonging to $W_k$) to the vertices of $V_{k+1}$ is in $G$,
3. connecting the vertices of $W_k - V_k$ to the vertices of $V_{k+1}$ so to make the existing ordering of vertices in $W_k$ decreasing in the subgraph induced by $W_k$ and $V_{k+1}$. More formally, whenever vertex $w_{k,i}$ in $W_k - V_k$ is connected to a vertex $v$ in $V_{k+1}$, every vertex $w_{k,j}$ with $j < i$ must be connected to $v$ too.

According to (2) and (3) the subgraph of $H'$ induced by $W_k$ and $V_{k+1}$ is a chain graph. We denote this subgraph by $G'$. Clearly $H'$ contains $G$ as an induced subgraph. To extend $H'$ to a canonical graph containing $G$ we apply the induction hypothesis twice. First, we extend $G'$ to a canonical chain graph as described in Theorem 4. This will add $m$ new vertices to the $k+1$-th and $n_k$ new vertices to $k$-th layer of the graph. Then we extend the first $k$ layers to a canonical form. The resulting graph has $k + 1 \leq n$ layers with $n$ vertices in each layer.

**Corollary 2** The universal bipartite permutation graph $H_{n,n}$ is optimal in order.

In the rest of this section, we show that some similar results hold for unit interval graphs. Indeed, between bipartite permutation graphs and unit interval graphs there
is a close relation, which can be described as follows. Given a bipartite permutation graph \(G\) with layers \(D_0, D_1, \ldots, D_q\), replace each independent set \(D_j\) with a clique (in other words, connect every two vertices in \(D_j\)). In this way, we obtain a unit interval graph. On the other hand, every connected unit interval graph can be partitioned into layers each of which is a clique. More formally,

**Theorem 7** A connected graph \(G\) is unit interval if and only if the vertex set of \(G\) can be partitioned into cliques \(D_0, D_1, \ldots, D_q\) so that

(a) any two vertices in non-consecutive cliques are non-adjacent,

(b) any two consecutive cliques \(D_{j-1}\) and \(D_j\) induce the complement of a chain graph, denoted \(G_j\),

(c) for each \(j = 1, 2, \ldots, q-1\), there is an ordering of vertices in \(D_j\), which is decreasing for \(G_j\) and increasing for \(G_{j+1}\).

This theorem can be proven by analogy with Theorem 5, for the prove of which an intersection model of bipartite permutation graphs has been used. We advise the reader to use the intersection model of unit interval graphs and leave the proof of Theorem 7 as an exercise. The relation between bipartite permutation and unit interval graphs suggests a similar construction of universal unit interval graphs.

6 General Bipartite Graphs

Let \(D_{n_1,n_2}\) denote the set of all bipartite graphs \(G = (V_1, V_2, E)\) with parts of size \(|V_1| = n_1\) and \(|V_2| = n_2\). Also,

\[
D_n = \bigcup_{n_1 + n_2 = n} D_{n_1,n_2}, \quad D = \bigcup_{n=1}^{\infty} D_n.
\]

We will construct an \(n\)-universal bipartite graph \(UD_n\) in the following way. With each partition \(n = n_1 + n_2\) we associate a connected component \(UD_{n_1,n_2}\) of the graph \(UD_n\) which contains all graphs from \(D_{n_1,n_2}\) as induced subgraphs.

**Lemma 1** For a complete bipartite graph \(K_{n_1,n_2} = (V_1, V_2, E_K)\) \(||V_1| = n_1, |V_2| = n_2\) there exists a partition \(E_K = E_1 \cup^* E_2\) such that for all \(v \in V_i\) \(\deg_{E_i}(v) \leq \left\lceil \frac{n_1+n_2}{4} \right\rceil\) holds \((i = 1, 2)\).

**Proof.** Let us assume \(n_1 \leq n_2\). Then there exists a set of edges \(E_1 \subset E_K\) such that

- \(\deg_{E_1}(v) = \left\lceil \frac{n_1+n_2}{4} \right\rceil\) for all \(v \in V_1\)
- \(|\deg_{E_1}(w) - \deg_{E_1}(z)| \leq 1\) for all \(w, z \in V_2\).
Let \( E_2 = E_K \setminus E_1 \); then for \( v \in V_2 \) we have \( \deg_{E_2}(v) \leq n_1 - \left\lfloor \frac{n_1 + n_2}{n_2} \right\rfloor \), therefore it suffices to show \( n_1 - \left\lfloor \frac{n_1 + n_2}{n_2} \right\rfloor \leq \left\lfloor \frac{n_1 + n_2}{4} \right\rfloor \), or equivalently, \( n_1 - \left\lfloor \frac{n_1 + n_2}{n_2} \right\rfloor \leq n_1 \left\lfloor \frac{n_1 + n_2}{n_2} \right\rfloor \). By rearranging this we get \( n_1 n_2 \leq (n_1 + n_2) \left\lceil \frac{n_1 + n_2}{4} \right\rceil \), which is true since by the inequality between arithmetic and geometric means we have \( n_1 n_2 \leq \left(\frac{n_1 + n_2}{2}\right)^2 \leq (n_1 + n_2) \left\lceil \frac{n_1 + n_2}{4} \right\rceil \).

Now, using the notations from the above theorem, we define \( UD_{n_1,n_2} \) as follows:

- The vertex set of \( UD_{n_1,n_2} \) is \( U_1 \cup U_2 \), where \( U_i = \{(v,F) \mid v \in V_i, \ F \subset N_{E_i}(v)\} \)

- Two vertices \((v_1,F_1) \in U_1\) and \((v_2,F_2) \in U_2\) are adjacent in \( UD_{n_1,n_2} \) if and only if either \( v_1 \in F_2 \) or \( v_2 \in F_1 \).

Let us consider an arbitrary bipartite graph \( G = (V_1,V_2,E) \) in \( D_{n_1,n_2} \). Then (still using our previous notations) it is easy to verify the following:

**Proposition 1** Mapping a vertex \( v \in V_i \) to \( (v,N_{E_i}(v) \cap N_{E_i}(v)) \in U_i \ (i = 1,2) \) provides us with an embedding of \( G \) into \( UD_{n_1,n_2} \).

As an immediate corollary we obtain

**Theorem 8** The graph \( UD_n \) constructed above is an asymptotically optimal \( n \)-universal bipartite graph.

**Proof.** The universality of \( UD_n \) follows from the previous proposition. According to our construction

\[
|UD_n| = \sum_{n_1 + n_2 = n} |UD_{n_1,n_2}| \leq \sum_{n_1 + n_2 = n} 2^{\left\lceil \frac{n}{2} \right\rceil} n \leq (n^2 + n)2^{\frac{n}{2} + 1}.
\]

It is known (see e.g. [7]) that \( |D_n| = 2^{n^2/4 + o(n^2)} \), which implies asymptotic optimality.

**References**


