Combinatorial Patterns for Probabilistically Constrained Optimization Problems

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Problem Formulation

Probabilistically constrained programming problem

$$
\begin{aligned}
\min\ & g(x) \\
\text{subject to } & Ax \geq b \\
& \mathcal{P}\left(h_j(x) \geq \xi_j, j \in J\right) \geq p \\
& x \in \mathcal{R}_+ \times \mathcal{Z}_+
\end{aligned}
$$

with ξ having a multivariate probability distribution with finite support

$\rightarrow$ Prékopa (1990,1995); Sen (1992); Prékopa et al. (1998); Dentcheva et al. (2000); Ruszczyński (2002); Cheon et al. (2006); Lejeune, Ruszczyński (2007); Luedtke et al. (2007); Tanner, Ntaimo (2008)

Example

$$\min x_1 + 2x_2$$

$$\text{subject to } \mathcal{P}\left\{\begin{array}{l} 8 - x_1 - 2x_2 \geq \xi_1 \\ 8x_1 + 6x_2 \geq \xi_2 \end{array}\right\} \geq 0.7$$

$$x_1, x_2 \geq 0$$

	k	ω_1^k	ω_2^k	$F(\omega^k)$
Set of realizations $\omega^k \in \Omega$	1	6	3	0.2
	2	2	3	0.1
	3	1	4	0.1
	4	4	5	0.3
	5	3	6	0.3
	6	4	6	0.5
	7	6	8	0.7
	8	1	9	0.2
	9	4	9	0.7
	10	5	10	0.8

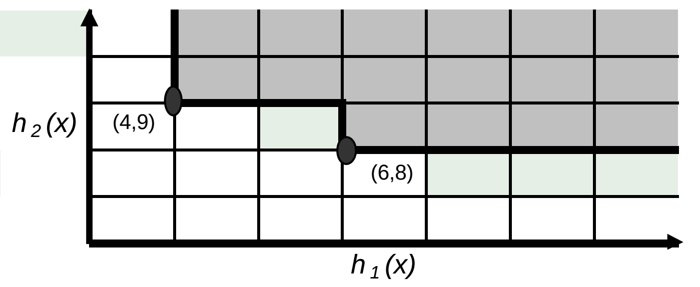
with $p_k = 0.1, k = 1, \ldots, 10.$

Example

Feasibility set is the union of the two following polyhedra:

- $S_1 = \{(x_1, x_2) \in \mathcal{R}^2_+ : 8 - x_1 - 2x_2 \geq 6,\ 8x_1 + 6x_2 \geq 8\}$,
- $S_2 = \{(x_1, x_2) \in \mathcal{R}^2_+ : 8 - x_1 - 2x_2 \geq 4,\ 8x_1 + 6x_2 \geq 9\}$,

and is non-convex:



Could also be “disconnected” (Henrion, 2002).

Solution Methods

- *p*-efficiency concept (Prékopa, 1990): disjunctive problem:
  - Identification of finite, unknown number of *p*-efficient points
  - Enumerative algorithm (Prékopa, 1995; Prékopa et al., 1990; Beraldi, Ruszczyński, 2002; Lejeune, 2008) or optimization-based generation (Lejeune, Noyan, 2009)
  - Convexification - cone generation algorithm (Dentcheva et al., 2001)
  - Column generation algorithm (Lejeune, Ruszczyński, 2007)
- Scenario approach
  - List possible realizations of multivariate random vector
  - Associate a binary variable with each scenario
  - MIP formulation with cover constraint
  - Use of structural properties (Ruszczyński, 2002; Cheon et al., 2006; Luedtke et al., 2007)
- Robust approach
  - Derivation of conservative and convex approximations (Calafiore, Campi, 2005; Nemirovski, Shapiro, 2005, 2006)

p-Efficiency

Definition (Prékopa, 1990)

Let $p \in [0, 1]$.
$v \in \mathcal{R}^n$ is a p-efficient point of the discrete probability distribution F if:

$$F(v) \geq p\ , \quad \text{and}$$
$$\text{there is no} \quad v' \leq v, v' \neq v \text{ such that } F(v') \geq p\ .$$

Identification of finite, unknown number of p-efficient points
Disjunctive problem

$$\begin{aligned} \min\ & g(x) \\ \text{subject to } & Ax \geq b \\ & h(x) \in \bigcup_{e \in S^p} K^e \\ & x \in \mathcal{R} \times \mathcal{Z} \end{aligned}$$

where $$K^e = v^e + \mathcal{R}_+,\ v^e \in S^p$$
is the cone associated with v^e, S^p is the set of p-efficient points.

p-efficiency

MIP reformulation

$$\min g(x)$$
$$\text{subject to } Ax \geq b$$
$$h_j(x) \geq \theta^e \cdot v_j^e,\ j \in J\ e \in S^p$$
$$\sum_{e \in S^p} \theta^e \geq 1$$
$$\theta \in \{0, 1\}$$
$$x \in \mathcal{R} \times \mathcal{Z}$$

Convexification

$$\min g(x)$$
$$\text{subject to } Ax \geq b$$
$$h_j(x) \geq \sum_{e \in S^p} \lambda^e \cdot v_j^e,\ j \in J\ e \in S^p$$
$$\sum_{e \in S^p} \lambda^e = 1$$
$$\lambda^e \in R_+$$
$$x \in \mathcal{R} \times \mathcal{Z}$$

Scenario Approach

- List possible realizations ξ^s of the multivariate random vector
- Associate a binary variable θ^s with each scenario s:

$$\theta^s = \begin{cases} 0 & \text{if all constraints in } s \text{ are satisfied} \\ 1 & \text{otherwise} \end{cases}$$

- MIP reformulation with cover constraint

$$\begin{aligned} \min\ & g(x) \\ \text{subject to}\ & Ax \geq b \\ & h_j(x) \geq \xi_j^s \cdot (1 - \theta^s), \quad j \in J, \forall s \\ & \sum_s p_s \cdot \theta^s \leq 1 - p \\ & \theta^s \in \{0, 1\}, \qquad \forall s \\ & x \in \mathcal{R} \times \mathcal{Z} \end{aligned}$$

with p_s = probability of scenario s

Structure

Solution framework based on combinatorial pattern theory:

$$\mathcal{P}\left(h_j(x) \geq \xi_j, j \in J\right) \geq p$$

- Binarization of probability distribution F
- Representation of combination (F, p) of probability distribution F and probability level p as partially defined Boolean function (pdBf)
- Compact extension
- Optimization-Based generation of combinatorial patterns
- Derivation of disjunctive normal form (DNF) representing sufficient conditions for probabilistic constraint to hold
  - Integrated DNF generation
  - Sequential DNF generation
  - Deterministic reformulations and solution
  - Concurrent pattern generation and solution

Numerical implementation

Conclusion

p-Sufficient and p-Insufficient Realizations

Definition (p-Sufficient Realization)

A realization ω^k is p-sufficient if $\mathcal{P}(\xi \leq \omega^k) = F(\omega^k) \geq p$ and is p-insufficient if $F(\omega^k) < p$.

Corollary

The satisfaction of the $|J|$ requirements

$$h_j(x) \geq \omega_j^k,\ j \in J$$

defined by a p-sufficient realization ω^k allows attainment of probability level p.

Partition

Partition of Ω with Boolean parameter $\mathcal{I}^k$

$$\mathcal{I}^k = \begin{cases} 1 & \text{if } F(\omega^k) \geq p \;\rightarrow\; p-\text{sufficient realization} \\ 0 & \text{otherwise} \;\rightarrow\; p-\text{insufficient realization} \end{cases}.$$

Example

	k	ω_1^k	ω_2^k	$\mathcal{I}^k$
Set Ω^- of *p*-insufficient realizations	1	6	3	0
	2	2	3	0
	3	1	4	0
	4	4	5	0
	5	3	6	0
	6	4	6	0
	8	1	9	0
Set Ω^+ of *p*-sufficient realizations	7	6	8	1
	9	4	9	1
	10	5	10	1

Binarization of Probability Distribution

- Introduction of binary attributes β_{ij}^k for each $\omega^k \in \Omega$
- Definition of their value with respect to cut points c_{ij}

$$\beta_{ij}^k = \begin{cases} 1 & \text{if } \omega_j^k \geq c_{ij} \\ 0 & \text{otherwise} \end{cases} \quad , i = 1, \ldots, n_j, j \in J$$

with

$$c_{i'j} < c_{ij} \;\Rightarrow\; \beta_{ij}^k \leq \beta_{i'j}^k \quad \text{for any} \quad i' < i,\; j \in J,$$

and C is the set of cut points: $|C| = \sum_{j \in J} n_j$.

Each numerical realization $\omega^k, k \in \Omega$ is mapped to a binary vector:

$$\beta^k = \left[\beta_{11}^k, \beta_{21}^k, \ldots, \beta_{ij}^k, \ldots\right]$$

Associating β^k with $\mathcal{I}^k$ provides a pdBf representation of (F, p)

Example

$$C = \{c_{11} = 5;\ c_{12} = 4;\ c_{22} = 6;\ c_{32} = 10\}$$

k	β^k_{11}	β^k_{12}	β^k_{22}	β^k_{32}	$\mathcal{I}^k$	
1	1	0	0	0	0	
2	0	0	0	0	0	
3	0	1	0	0	0	
4	0	1	0	0	0	Binary Image Ω^-_B of Ω^-
5	0	1	1	0	0	
6	0	1	1	0	0	
8	0	1	1	0	0	
7	1	1	1	0	1	
9	0	1	1	0	1	Binary Image Ω^+_B of Ω^+
10	1	1	1	1	1	

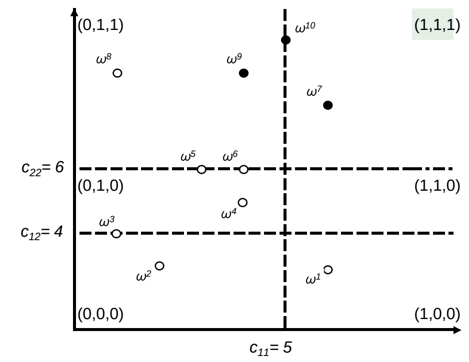
Definition of Set of Cut Points

Objective: define conditions for $\mathcal{P}\left(h_j(x) \geq \xi_j, j \in J\right) \geq p$ to hold

Set of cut points cannot be defined arbitrarily

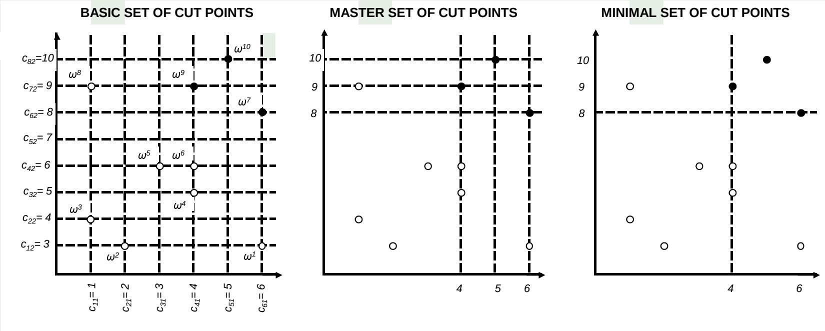
Example

$C = \{c_{11} = 5;\ c_{12} = 4;\ c_{22} = 6\}$



- Preserves the *disjointedness* between Ω^+ and Ω^-
- *Consistency* of the set of cut points:
  - *basic*: immediate: $C = \{c_{ij} : c_{ij} = \omega_j^k,\ j \in J, k \in \Omega\}$
  - *master*: polynomial-time algorithm
  - *minimal*: set covering formulation

Example



Necessary Conditions

Consistency of set of cut points is not sufficient.

Example

Consider the minimal set of cut points: $C = \{c_{11} = 4;\ c_{12} = 8\}$.

$$\left\{ \begin{array}{l} \omega_1^k \geq 4 \\ \omega_2^k \geq 8 \end{array} \right. \quad \Rightarrow \quad \begin{array}{l} \text{Satisfied by each} \quad \omega^k \in \Omega^+ \\ \text{Not satisfied by any } \omega^k \in \Omega^- \end{array}$$

$$\mathcal{P}\left\{ \begin{array}{l} 8 - x_1 - 2x_2 \geq \xi_1 \\ 8x_1 + 6x_2 \geq \xi_2 \end{array} \right\} \geq 0.7 \not\Leftrightarrow \left\{ \begin{array}{l} 8 - x_1 - 2x_2 \geq 4 \\ 8x_1 + 6x_2 \geq 8 \end{array} \right.$$

Set $8 - x_1 - 2x_2 = 4$ and $8x_1 + 6x_2 = 8 : \mathcal{P}(4 \geq \xi_1, 8 \geq \xi_2) = 0.5 < p$

Consistency does not guarantee *exact* representation of all the p-sufficient realizations:

$$\omega_j^k = \bigvee_{i=1,\ldots,n_j} \beta_{ij}^k \cdot c_{ij} \ , \ j \in J, \omega^k \in \Omega^+$$

Sufficient Conditions

$$\mathcal{P}\left(h_j(x) \geq \xi_j, j \in J\right) \geq p \text{ if } P\left(h_j(x) \geq \xi_j\right) \geq p,\ j \in J$$

Definition

A sufficient-equivalent set of cut points C^E comprises a cut point c_{ij} for any value ω_j^k taken by any of the p-sufficient realizations on any of the marginals j:

$$C^E = \{c_{ij} : F_j(c_{ij}) \geq p,\ i = 1, \ldots, n_j, j \in J, k \in \Omega\}$$

Allows the *exact* representation of all the p-sufficient realizations, and is thus consistent.

Example

$$C^E = \{\underbrace{4, 5, 6}_{\xi_1};\ \underbrace{8, 9, 10}_{\xi_2}\}.$$

Coincides here with master set of cut points.

Extension

- Objective: Simple and compact representation of (F, p)
- Definition: f is an extension of pdBf $g(\Omega_B^+, \Omega_B^-)$ if:

$$\Omega_B^+ \subseteq \Omega_B^+(f) \text{ and } \Omega_B^- \subseteq \Omega_B^-(f)$$

- Existence: Boolean extension f exists if and only if $\Omega_B^+ \bigcap \Omega_B^- = \emptyset$
- Description: Disjunctive normal form
  - Binary mapping of realization: $\omega^k \to \beta^k = \left[\beta_{11}^k, \ldots, \beta_{ij}^k, \ldots\right]$
  - Set of binary images: $\Omega_B = \Omega_B^+ \bigcup \Omega_B^-, \; \Omega_B^+ \bigcap \Omega_B^- = \emptyset$
  - Literals $\beta_{ij}, \; \bar{\beta}_{ij}$
  - *Pattern*: *term* (*clause*): $t = \bigwedge_{ij \in P_t} \beta_{ij} \bigwedge_{ij \in N_t} \bar{\beta}_{ij}, \quad P_t \bigcap N_t = \emptyset$ with *coverage* condition
  - Term *covers* a realization ω^k if : $t(\omega^k) = 1 = \bigwedge_{ij \in P_t} \beta_{ij}^k \bigwedge_{ij \in N_t} \bar{\beta}_{ij}^k$,
  - *Degree* of a term: number of literals: $d = |P_t| + |N_t|$,
  - *Disjunctive Normal Form*: $f = \bigvee_{v \in V} t_s$.

Properties

Any Boolean extension of a consistent pdBf representing (F, p) is a:

- positive monotone,
- Horn,
- threshold

Boolean function.

Rationale for Optimization-Based Generation

- Patterns included in DNF representing (F, p) are of degree at least equal to $|J|$.
  Recall: $\mathcal{P}\left(h_j(x) \geq \xi_j, j \in J\right) \geq p$
- Patterns often generated though term enumeration methods (Boros et al., 1997, 2000; Alexe, Hammer, 2006, 2007; Torvik, Triantaphyllou, 2006)
- Needs considering $\sum_{d'=1}^{d} 2^{d'} \binom{n}{d'}$ terms for patterns of degree d
- Very efficient except for patterns of high degree (larger than 4) (Boros et al., 1997, 2000; Ryoo, 2006, 2008)

Optimization-Based Generation of Patterns - IP I

Consider a sufficient-equivalent set of cut points and pdBf for (F, p).

$$\begin{array}{lll}
IP\ I & z = \min \sum_{k \in \Omega_B^+} y^k & \\
\text{subject to} & \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} + \sum_{e=1}^{n} \bar{\beta}_{ij}^k \bar{u}_{ij} + n y^k \geq d, & k \in \Omega_B^+ \\
& \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} + \sum_{e=1}^{n} \bar{\beta}_{ij}^k \bar{u}_{ij} \leq d - 1, & k \in \Omega_B^- \\
& u_{\eta_j^k j} \geq 1 - b^k, & k \in \Omega_B^+, j \in J \\
& \sum_{k \in \Omega_B^+} b_k = |\Omega_B^+| - 1 & \\
& u_{ij} + \bar{u}_{ij} \leq 1, & i = 1, \ldots, n_j, j \in J \\
& \sum_{j \in J} \sum_{i=1}^{n_j} (u_{ij} + \bar{u}_{ij}) = d & \\
& 0 \leq b^k \leq 1, & k \in \Omega_B^+ \\
& |J| \leq d \leq 2n & \\
& u_{ij}, \bar{u}_{ij} \in \{0, 1\}, & i = 1, \ldots, n_j, j \in J \\
& y^k \in \{0, 1\}, & k \in \Omega_B^+
\end{array}$$

Properties

Theorem (Pattern Generation - IP I)

IP I:
(i) is always feasible;
(ii) has an upper bound equal to $|\Omega_B^+| - 1$*; and*
(iii) any of its feasible solutions $(\mathbf{u}, \mathbf{y}, \mathbf{d}, \mathbf{b})$ *defines a p-sufficient pattern*

$$t = \bigwedge_{\substack{\mathbf{u_{ij}}=\mathbf{1} \\ i=1,\ldots,n_j, j\in J}} \beta_{ij} \bigwedge_{\substack{\bar{\mathbf{u}}_{\bar{ij}}=1 \\ i=1,\ldots,n_j, j\in J}} \bar{\beta}_{ij} \quad \textit{of degree } d \textit{ and coverage } \left(\left|\Omega_B^+\right| - \mathbf{z}\right).$$

Remarks:

- Complexity: $2n + |\Omega^+|$ integer variables
- Increases with number of cut points and *p*-sufficient realizations
- Number of *p*-sufficient realizations is a decreasing function of *p*
- Does not need to be solved to optimality
- Optimal solution is a *p*-sufficient strong pattern (Hammer et al., 2004)

Pattern Derivation

Definition (Hammer et al., 2004)

A pattern is *prime* if the removal of any one of its literals results in the coverage of a realization of opposed "sign".

Observation:

ω_j is positive monotone in F:

$$\mathcal{P}(\xi_j \leq \omega_j^k) \leq \mathcal{P}(\xi_j \leq \omega_j^{k'}) \text{ for } \omega_j^k \leq \omega_j^{k'}, j \in J$$

β_{ij} is positive monotone in the Boolean extension f:

$$f\left(\beta_{11}, \beta_{21}, \ldots, \beta_{i-1j}, 0, \beta_{i+1j}, \ldots\right) \leq f\left(\beta_{11}, \beta_{21}, \ldots, \beta_{i-1j}, 1, \beta_{i+1j}, \ldots\right)$$

$\Rightarrow$ Prime patterns included in a DNF f representing (F, p)

- do not include complemented literals: monotonicity property of Boolean variable (Boros et al., 2000)
- one uncomplemented literal per component ξ_j

$$
\begin{array}{lll}
\textit{IP II} & z = \min \sum_{k \in \Omega_B^+} y^k & \\
\text{subject to} & \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} + y^k \geq |J|, & k \in \Omega_B^+ \\
& \sum_{j \in J} \sum_{i=1}^{n_j} \beta_{ij}^k u_{ij} \leq |J| - 1, & k \in \Omega_B^- \\
& u_{\eta_j^k j} \geq 1 - b^k, & k \in \Omega_B^+, j \in J \\
& \sum_{k \in \Omega_B^+} b_k = |\Omega_B^+| - 1 & \\
& \sum_{i=1}^{n_j} u_{ij} = 1, & j \in J \\
& 0 \leq b^k \leq 1, & k \in \Omega_B^+ \\
& u_{ij} \in \{0, 1\}, & j \in J, i = 1, \ldots, n_j \\
& 0 \leq y^k \leq |J|, & k \in \Omega_B^+
\end{array}
$$

Properties

Theorem (Pattern Generation - IP II)

IP II:
(i) is always feasible, and
(ii) any of its feasible solutions $(\mathbf{u}, \mathbf{y}, \mathbf{b})$ *defines a p-sufficient pattern*

$$t = \bigwedge_{\substack{\mathbf{u_{ij}=1} \\ j \in J, i=1,\ldots,n_j}} \beta_{ij}$$

of degree $|J|$.

Comparison:

- IP I: $2n + |\Omega^+|$ integer and $|\Omega^+| + 1$ continuous variables
- IP II: n integer and $2|\Omega^+|$ continuous variables

DNF Derivation - Integrated Approach: IP III

$$\textit{IP III} \qquad \max \sum_{s=1}^{Q} y_s$$

subject to

$$
\begin{aligned}
&\sum_{i=1}^{n_j}\sum_{j\in J} \beta_{ij}^k u_{ij,s} + |J| y_s^k \geq |J|, && k \in \Omega_B^+, s = 1,\ldots,Q \\
&r_s^k \geq y_s^k, && k \in \Omega_B^+, s = 1,\ldots,Q \\
&\sum_{s=1}^{Q} r_s^k \leq Q - 1, && k \in \Omega_B^+ \\
&r_s^k \geq y_s, && k \in \Omega_B^+, s = 1,\ldots,Q \\
&u_{\eta_j^k j,s} \geq 1 - b_s^k, && k \in \Omega_B^+, j \in J, s = 1,\ldots,Q \\
&y_s = \sum_{k\in\Omega_B^+} b_s^k + 1 - |\Omega_B^+|, && s = 1,\ldots,Q \\
&\sum_{i=1}^{n_j} u_{ij,s} = 1, && j \in J, s = 1,\ldots,Q \\
&0 \leq b_s^k \leq 1, && k \in \Omega_B^+, s = 1,\ldots,Q \\
&0 \leq r_s^k \leq 1, && k \in \Omega_B^+, s = 1,\ldots,Q \\
&0 \leq y_s \leq 1, && s = 1,\ldots,Q \\
&y_s^k \in \{0,1\}, && k \in \Omega_B^+, s = 1,\ldots,Q \\
&u_{ij,s} \in \{0,1\}, && i = 1,\ldots,n_j, j \in J, s = 1,\ldots,Q
\end{aligned}
$$

Properties

Theorem (Disjunctive Normal Form Model)

Any feasible solution $(\mathbf{u}, \mathbf{y}, \mathbf{r}, \mathbf{b})$ *of IP III defines a DNF*

$$f = \bigvee_{\mathbf{y_s}=\mathbf{0}} t_s$$

including a set of patterns $\mathcal{Q} = \{t_s : \mathbf{y_s} = 0, \forall s\}$*:*

i) covering all p-sufficient realizations: $f\left(\omega^k\right) = 1,\ k \in \Omega_B^+$*, and*

ii) defining the sufficient conditions for $\mathcal{P}\left(h_j(x) \geq \xi_j, j \in J\right) \geq p$ *to hold.*

Remarks:

- each t_s in f is of degree $|J|$;
- each t_s in f has coverage $|\Omega^+| - \sum_{k \in \Omega_B^+} \mathbf{y_s^k}$;
- the optimal solution of IP III defines an irredundant DNF.

DNF Derivation - Sequential Approach

- Iterative procedure
- Ordering of *p*-sufficient realizations with respect to their cumulative probability
- Concept of *maximum* positive pattern (Hammer, Bonates, 2006)

Definition

The maximum *p*-sufficient ω^k-pattern is the pattern covering ω^k which has the largest coverage.

- Differences with integrated approach:
  - Disjunctive normal form is not necessarily minimal
  - Solution of a finite sequence of LP problems

Deterministic Reformulation I

f: DNF defining sufficient conditions for satisfiability of

$$\mathcal{P}(h_j(x) \geq \xi_j,\ j \in J) \geq p$$

$$\begin{aligned} \min\ & g(x) \\ \text{subject to } & Ax \geq b \\ & f(h(x)) \geq 1 \\ & x \in \mathcal{R}_+ \times \mathcal{Z}_+ \end{aligned}$$

$$f(h(x)) = \bigvee_{v=1,\ldots,V} t_v(h(x)) \geq 1 \;\Leftrightarrow\; \sum_{v=1}^{V} t_v(h(x)) \geq 1$$

Deterministic Reformulation II

$$f(h(x)) = 1 \Leftrightarrow \bigvee_{v=1,\ldots,V} t_v(h(x)) \geq 1 \Leftrightarrow \sum_{v=1}^{V} t_v(h(x)) \geq 1$$

$$t_v = \bigwedge_{ij \in L_v} \beta_{ij} : t_v(h(x)) = 1 \Rightarrow h_j(x) \geq c_{ij},\ ij \in L_v$$

$$\gamma_v = \begin{cases} 0, & \text{if all conditions defined by } t_v \text{ are satisfied} \\ 1, & \text{otherwise} \end{cases}$$

$$\begin{cases} \gamma_v + t_v(h(x)) = 1,\ v = 1, \ldots, V \\ \sum_{v=1}^{V} \gamma_v \leq V - 1 \end{cases} \Leftrightarrow \begin{cases} h_j(x) + M\gamma_v \geq c_{ij},\ ij \in L_v \\ \sum_{v=1}^{V} \gamma_v \leq V - 1 \end{cases}$$

Concurrent Generation and Solution

$$
\begin{aligned}
\min\ & g(x) \\
\text{subject to } & Ax \geq b \\
& \sum_{i=1}^{n_j} u_{ij} = 1, && j \in J \\
& u_{\eta_j^k j} \geq 1 - b^k, && k \in \Omega_B^+, j \in J \\
& \sum_{k \in \Omega_B^+} b^k \leq |\Omega_B^+| - 1 \\
& h_j(x) \geq u_{ij} \cdot c_{ij}, && i = 1, \ldots, n_j, j \in J \\
& 0 \leq b^k \leq 1, && k \in \Omega_B^+ \\
& u_{ij} \in \{0, 1\}, && i = 1, \ldots, n_j, j \in J \\
& x \in \mathcal{R}_+ \times \mathcal{Z}_+
\end{aligned}
$$

Optimal solution $(\mathbf{x}^*, \mathbf{u}^*, \mathbf{b}^*)$ defines a p-sufficient pattern $t = \bigwedge_{\substack{\mathbf{u}_{\mathbf{ij}}^* = \mathbf{1} \\ i=1,\ldots,n_j, j \in J}} \beta_{ij}$

representing the *minimal* conditions for $\mathcal{P}(h_j(x) \geq \xi_j,\ j \in J) \geq p$ to hold.

Numerical Implementation

Stochastic cash matching (Dentcheva et al., 2004; Henrion, 2004)

$$\max \sum_{i=1}^{n} \left(a_{i|J|} - p_i\right) x_i$$

$$\text{subject to } \mathcal{P}(K + \sum_{i=1}^{n} (a_{ij} - p_i) x_i \geq \xi_j,\ j \in J) \geq p$$

$$x \in \mathcal{R}_+$$

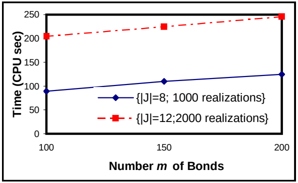
Data: face value, yield structure, maturity of more than 200 bonds
Sources: Center for Research and Security prices (CRSP); Mergent Fixed Income Securities Database (FISD).
Generation of 32 problem instances differing along:

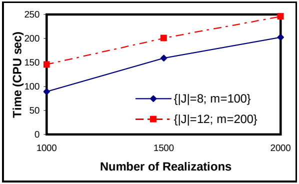
- number (M = 150, 200) of bonds
- length of planning horizon (i.e., dimensionality: $|J|$ = 8, 12 of the random vector ξ)
- value (p = 0.8, 0.85, 0.9, 0.95) of enforced probability level
- number (Ω = 1000, 2000) of realizations

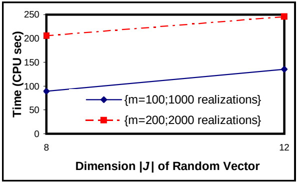
Sequential procedure

AMPL modeling, 11.1 solver for MIP

		p							
		0.8		0.85		0.9		0.95	
		$\|\Omega\|$							
M	$\|J\|$	1000	2000	1000	2000	1000	2000	1000	2000
150	8	305.0	369.3	145.3	239.2	68.9	94.3	14.2	20.9
150	12	299.3	421.7	176.2	295.9	87.9	109.9	23.9	35.8
200	8	341.9	375.9	146.3	248.9	71.9	100.3	12.2	24.9
200	12	362.2	418.1	172.9	299.1	92.2	103.9	31.8	49.2







Conclusions and Extensions

- Novel methodology for probabilistically constrained problems
- Derivation of combinatorial patterns and DNfs representing sufficient conditions for attainment of prescribed probability level
  - Binarization of probability distribution
  - Representation of (F, p) as pdBf
  - Extension of pdBf
  - Optimization-based derivation of patterns and DNFs
  - Deterministic reformulation
- Combinatorial pattern take into account "interactions" between components ξ_j of ξ on satisfiability of joint probabilistic constraint
- Commonalities with Logical Analysis of Data (Hammer, 1986; Crama et al., 1988; Boros et al., 1997, 2000)
- Numerical implementation
- Extensions possible to:
  - problems with random technology matrix
  - continuous probability distributions approximated by samples
  - two-stage stochastic problems.