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On the analytical–numerical valuation of the American option

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Abstract. The paper further develops, both from the theoretical and numerical points of view the analytic valuation of the American options, initiated by Geske and Johnson (1984) for the American put with no dividend. We present and prove closed form formulas for the Bermudan put and call, with dividend, paid continuously at a constant rate, where a general number and not necessarily equal length intervals subdivide the time. Based on the obtained formulas and recent, efficient numerical integration techniques, to obtain values of the multivariate normal c.d.f., the Bermudan put and call option values are evaluated for up to twenty subdividing intervals. The sequences of option values are smoothed by sums of exponential functions and the latters are used to predict the values of the American options. Numerical results are presented and compared with those, published in the literature. It is shown that the binomial method systematically overestimates the option price and so do other methods we have looked at, according to our results. Some properties of Richardson approximation are explored.

1 Introduction

Over the past thirty years a large number of papers have been published on the valuation or pricing of the American options. Much less, still not small is the number of those papers that deal with numerical methods to calculate/approximate the option values. Excellent summarizing papers by Broadie and Detemple (1996, 2004) and Ju (1998) provide us with information and insight into the various methodologies.

To cite a few of the most prominent ones we mention the finite difference method of Brennan and Schwartz (1977, 1978) that solves approximately the Black–Scholes–Merton PDE, the binomial pricing method of Cox, Ross and Rubinstein (1979), the analytic method of Geske and Johnson (1984), and other methods by Barone-Adesi and Whaley (1987), Kim (1990) and Broadie and Detemple (1996).

The paper that serves as starting point of our investigation is the one by Geske and Johnson (1984). These authors look at the Bermudan put option on an asset that does not pay dividend, write up a formula for the option value, where time is subdivided into n equal parts, compute numerically P_1, P_2, P_3, P_4 and then apply the Richardson extrapolation to approximate the value of the American put. Looking at the numerical results, the method seems to be efficient at least on the instances presented in the paper. However, rigorous mathematical proofs for the option price formulas are not supplied and, on the other hand, the Richardson approximation is used for a small number of subdividing intervals.

The purpose of our paper is to present the closed form formulas for the prices of the Bermudan put and call options, where the asset pays continuously dividend with constant rate, provide exact mathematical proofs for them, numerically calculate the option prices for up to 20 subdivisions of the time and predict the prices of the American options. The formulas are presented for the general case, where the subdividing intervals are not necessarily equal.

The celebrated Black–Scholes–Merton (BSM) formulas for the prices of the European call and put options are:

$$c = Se^{-D(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2) \tag{1.1}$$

$$p = Xe^{-r(T-t)}N(-d_2) - Se^{-D(T-t)}N(-d_1), \tag{1.2}$$

where

$$d_1 = \frac{\ln\frac{S}{X} + \left(r - D + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{1.3}$$

$$d_2 = \frac{\ln\frac{S}{X} + \left(r - D - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{1.4}$$

In this formula t designates the current time, T the expiration, S the current price of the stock, r the rate of interest (assumed to be constant), D the rate of dividend, σ^2 the variance of an underlying geometric Brownian motion that describes the time variation of the stock price, and $N(\cdot)$ the c.d.f. of the univariate normal distribution. The geometric Bwownian motion of the stock price process is of the form

$$S(t) = S(0)e^{\sigma B(t)+\mu t}, \qquad t \geq 0, \tag{1.5}$$

where $\sigma > 0$ is the already introduced constant, $\mu = r - \dfrac{\sigma^2}{2}$ and $B(t)$, $t \geq 0$ is a standard Brownian motion process.

The price dynamics is characterized by the stochastic differential equation:

$$\frac{\mathrm{d}S(t)}{S(t)} = \sigma \mathrm{d}B(t) + (r - D)\mathrm{d}t. \tag{1.6}$$

The assumption that the stock price performs a geometric Brownian motion is generally attributed to Black and Scholes (1973) and Merton (1973). There are, however, precursors of the model as well as of the formulas (1.1)–(1.2). We only mention that Bachelier (1900) was the first to introduce Brownian motion into financial theory and Sprenkle (1961), Boness (1964) and Samuelson (1965) already worked with the multiplicative Brownian motion hypothesis. For references and a good summary of the early results the reader is referred to Briys, Bellalah, Minh Mai and de Varenne (1998).

Black and Scholes (1973) and Merton (1973) derived the parabolic PDE, for the case of $D = 0$, valid for the value function $V = V(t)$ of any derivative:

$$\frac{\partial V}{\partial t} + (r - D)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV. \tag{1.7}$$

Formulas (1.1)–(1.4) have been obtained by the solutions of the equation (1.7), with the boundary conditions $V = [S - X]_+$, $V = [X - S]_+$, assumed to hold at time T, for the European call and put options, respectively.

The paper is organized as follows. In Section 2 we present dynamic programming recursion formulas for the Bermudan put and derive some properties of the value functions involved. In Section 3 we present, with detailed proofs, the option pricing formulas for the Bermudan put with dividend. The formulas for the Bermudan call are presented in Section 4. In Section 5 we define the value of the American option as the limit of a sequence of Bermudan option values and establish the convergence of the value sequences, given by our formulas. In Section 6 we briefly describe the numerical integration techniques used in the paper and present numerical results. In Section 7 we point out that the binomial pricing method systematically overestimates the option price. In Section 8 some remarks are made concerning the Richardson approximation. Finally in Section 9 we summarize the conclusions.

In what follows we make use of a formula in connection with multivariate normal integrals. A multivariate normal distribution is called standard if all of its univariate marginal distributions are standard.

If $R = (\rho_{ij})$ is the correlation matrix of the random variables involved, then the c.d.f. of the n-variate standard normal distribution is designated by $\Phi(x_1, \ldots, x_n; R)$ or $\Phi(x; R)$, where $x^T = (x_1, \dots, x_n)$. If $X_1, \dots, X_n$ have this joint distribution, then the conditional distribution function of $X_2, \dots, X_n$, given $X_1 = u$ is equal to (see, e.g., T.W. Anderson, 1957):

$$\Phi\left(\frac{x_2 - \rho_{21}u}{\sqrt{1-\rho_{21}^2}}, \dots, \frac{x_n - \rho_{n1}u}{\sqrt{1-\rho_{n1}^2}}; R_1\right), \tag{1.8}$$

where R_1 is the $(n-1) \times (n-1)$ correlation matrix with entries

$$\frac{\rho_{ij} - \rho_{1i}\rho_{1j}}{\sqrt{1-\rho_{1i}^2}\sqrt{1-\rho_{1j}^2}}, \qquad 2 \le i, j \le n. \tag{1.9}$$

This implies that

$$\Phi(x_1, \dots, x_n; R) = \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho_{21}u}{\sqrt{1-\rho_{21}^2}}, \dots, \frac{x_n - \rho_{n1}u}{\sqrt{1-\rho_{n1}^2}}; R_0\right) \varphi(u)\, \mathrm{d}u, \tag{1.10}$$

where $\varphi(\cdot)$ is the p.d.f. of the univariate standard normal distribution. Formula (1.10) is called reduction formula. It enables the calculation of the values of the n-variate standard normal c.d.f. by the use of the values of $n-1$-variate ones. For results in connection with the normal distribution the reader is referred to Tong (1990).

2 Dynamic programming recursion for the value of the Bermudan put option

We assume that the price process is a multiplicative Brownian motion process, i.e., it has the form (1.5). We use risk neutral valuation which means we assume that $\mu = r - \frac{\sigma^2}{2}$. We also assume that the asset pays dividend continuously, at a constant rate D.

Let us subdivide the time interval $[t, T]$ into n parts and let the subdividing points be $t_1, \dots, t_{n-1}$, where $t < t_1 < \dots < t_{n-1} < T$. Introduce the notations: $t_0 = t$, $t_n = T$, $\Delta t_i = t_i - t_{i-1}$, $i = 1, \dots, n$. If the only possible exercise times are $t_0, t_1, \dots, t_n$, then the option is called Bermudan. Its value can be taken as an approximate value of the American option. Let p, P designate the European and American put prices, and c, C the European and American call prices, respectively. In case of n subdividing intervals the Bermudan option prices are designated by P_n and C_n, respectively.

At any time t the option (spot) payoff is defined as the value $[X-S]_+$, where S is the spot price of the asset. Assume that the Bermudan option will be exercised whenever the spot payoff becomes at least as large as the current value of the option. If, at time t_i, $i = 1, \dots, n$, the spot payoff becomes at least as large as the current value of the option, then S is called a critical price corresponding to t_i. At time t_n the value of the option is 0, hence the critical price is equal to X. Let $V_n(S), V_{n-1}(S), \dots, V_0(S)$ designate the option values corresponding to $t_n, t_{n-1}, \dots, t_0$, respectively, where in each $V_i(S)$, the value S represents the spot price of the asset. It is a variable and $V_i(S)$ is its function which provides us with the option values for all possible values of $S > 0$. For $S = 0$ the above functions are defined by continuity. In view of our assumptions, we have the following recursion formulas:

$$\begin{aligned} V_n(S) &= [X - S]_+ \\ V_i(S) &= \max\left([X-S]_+, e^{-r\Delta t_{i+1}} E(V_{i+1}(Se^{-D\Delta t_{i+1}} Y_{i+1}))\right) \\ &\quad i = 0, \ldots, n-1, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \log Y_i &\sim N\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t_i, \sigma^2 \Delta t_i\right) \\ &\quad i = 1, \ldots, n. \end{aligned} \tag{2.2}$$

The price of the Bermudan option is $P_n = V_0(S)$. Let $V(t) = V_0(t)$.

Some properties of the $V_i(S)$ can immediately be derived. The function $V_n(S) = [S - X]_+$ is obviously continuous, decreasing in S for $S \geq 0$ and strictly decreasing in the interval $0 \leq S \leq X$.

Theorem 2.1 *The following assertions hold true*

(a) $V_i(0) = X$, $i = 0, \ldots, n$,

(b) *for every* $i = 0, \ldots, n-1$ *the function* $V_i(S)$ *is continuous and strictly decreasing for* $S \geq 0$.

Proof. Assertion (a) and the continuity of the functions $V_i(S)$ are simple facts. The rest of assertion (b) can be proved recursively.

For $i = n - 1$, $V_i(S)$ is the value of a European put option. It is well-known and also simple to check that for the option price given by (1.2)–(1.4) we have

$$\frac{\partial p}{\partial S} = N(d_1) - 1, \text{ for } S \geq 0.$$

It follows that the function

$$e^{-r\Delta t_n} E(V_n(Se^{-D\Delta t_n} Y_n))$$

is strictly decreasing in S. Its value at $S = 0$ is $e^{-r\Delta t_n} X$, hence there exists an $S = q_1$ such that $0 < q_1 < X$,

$$\begin{aligned} X - S &> e^{-r\Delta t_n} E(V_n(SY_n)) \quad \text{if} \quad S < q_1 \\ X - S &< e^{-r\Delta t_n} E(V_n(SY_n)) \quad \text{if} \quad S > q_1 \end{aligned}$$

and equality holds if $S = q_1$. This implies that $V_{n-1}(S)$ is strictly decreasing for $S \geq 0$. Continuing this way the assertion follows. □

The proof of Theorem 2.1 implies that for every $i = 1, \ldots, n-1$ there exists a unique S such that

$$X - S = e^{-r\Delta t_i} E(V_i(Se^{-D\Delta t_{i+1}} Y_i)). \tag{2.3}$$

Let q_{n-i} designate this value. Let $q_0 = X$ and call $q_0, \ldots, q_{n-1}$ critical prices.

THEOREM 2.2 *Suppose that $\Delta t_i = \Delta t = (T-t)/n$, $i = 1,\dots,n$. Then for any $S \geq 0$ we have the relation*

$$V_{i-1}(S) \geq V_i(S). \tag{2.4}$$

The inequality is strict if $S > q_{n-i+1}$ and equality holds if $S \leq q_{n-i+1}$. In the latter case $V_{i-1}(S) = V_i(S) = X - S$. In addition, we have the inequalities

$$q_0 = X > q_1 > \cdots > q_{n-1}. \tag{2.5}$$

PROOF. By (2.1) we have that $V_{n-1}(S) \geq V_n(S)$. This and the repeated applications of the equalities

$$V_i(S) = \max\left(X - S, e^{-r\Delta t}E(V_{i+1}(Se^{-D\Delta t_{i+1}}Y_{i+1}))\right)$$
$$V_{i-1}(S) = \max\left(X - S, e^{-r\Delta t}E(V_i(Se^{-D\Delta t_i}Y_i))\right)$$

prove (2.4).

By construction we have the inequality $V_{n-1}(S) > V_n(S)$ for $S > q_1$. If we use an inductive argument from i to $i-1$, then we can see that the assertion in connection with the strict inequality in (2.4) holds true. This, in turn, implies (2.5) and the theorem is proved. □

A subdivision $t_0 < t_1 < \cdots < t_n$ of the interval $[t_0, t_n]$ will be designated by τ letters. A subdivision is called equidistant if $t_i - t_{i-1} = \dfrac{t_n - t_0}{n}$, $i = 1,\dots,n$. If τ_1 and τ_2 are two equidistant subdivisions such that each point in τ_1 appears also in τ_2, then we write $\tau_1 \prec \tau_2$.

THEOREM 2.3 *If τ_1, τ_2 are two equidistant subdivisions and $\tau_1 \prec \tau_2$, then, the value of the Bermudan option corresponding to τ_1 is not greater than the value corresponding to τ_2.*

PROOF. Follows easily from the recursion (2.1). □

3 Formulas for the price of the Bermudan put option

Let us introduce the notations:

$$d_1(S,q,\tau) = \frac{\log\dfrac{S}{q} + \left(r + \dfrac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$
$$d_2(S,q,\tau) = \frac{\log\dfrac{S}{q} + \left(r - \dfrac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = d_1(S,q,\tau) - \sigma\sqrt{\tau}. \tag{3.1}$$

We have already defined $q_0 = X$, $q_1,\dots,q_{n-1}$. Now we give them new definitions and later on show that the two definitions provide us with the same numbers. First, for a given subdivision of the interval $[t,T]$, and given r, D, σ, $q_0 > q_1 > \cdots > q_{n-1}$ we formally write up the formulas of the function sequences $U_i(S)$, $W_i(S)$:

$$
\begin{aligned}
&U_i(S) \\
&\quad = e^{-r(t_{n-i+1}-t_{n-i})} N_1(-d_2(S, q_{i-1}, t_{n-i+1}-t_{n-i})) \\
&\qquad + e^{-r(t_{n-i+2}-t_{n-i})} N_2(d_2(S, q_{i-1}, t_{n-i+1}-t_{n-i}), \\
&\qquad - d_2(S, q_{i-2}, t_{n-i+2}-t_{n-i}); R^{(2)}) \\
&\qquad + e^{-r(t_{n-i+3}-t_{n-i})} N_3(d_2(S, q_{i-1}, t_{n-i+1}-t_{n-i}), d_2(S, q_{i-2}, t_{n-i+2}-t_{n-i}), \\
&\qquad - d_2(S, q_{i-3}, t_{n-i+3}-t_{n-i}); R^{(3)}) + \cdots \\
&\qquad + e^{-r(t_{n-i+h}-t_{n-i})} N_h(d_2(S, q_{i-1}, t_{n-i+1}-t_{n-i}), \ldots, d_2(S, q_{i-h+1}, t_{n-i+h-1}-t_{n-i}), \\
&\qquad - d_2(S, q_{i-h}, t_{n-i+h}-t_{n-h}); R^{(h)}) + \cdots \\
&\qquad + e^{-r(t_n-t_{n-i})} N_i(d_2(S, q_{i-1}, t_{n-i+1}-t_{n-i}), \ldots, d_2(S, q_1, t_{n-1}-t_{n-i}), \\
&\qquad - d_2(S, q_0, t_n-t_{n-i}); R^{(i)}), \\
&\qquad\quad i = 1, \ldots, n,
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
&W_i(S) \\
&\quad = e^{-D(t_{n-i+1}-t_{n-i})} N_1(-d_1(S, q_{i-1}, t_{n-i+1}-t_{n-i})) \\
&\qquad + e^{-D(t_{n-i+2}-t_{n-i})} N_2(d_1(S, q_{i-1}, t_{n-i+1}-t_{n-i}), -d_1(S, q_{i-2}, t_{n-i+2}-t_{n-i}); R^{(2)}) \\
&\qquad + e^{-D(t_{n-i+3}-t_{n-i})} N_3(d_1(S, q_{i-1}, t_{n-i+1}-t_{n-i}), d_1(S, q_{i-2}, t_{n-i+2}-t_{n-i}), \\
&\qquad - d_1(S, q_{i-3}, t_{n-i+3}-t_{n-i}); R^{(3)}) + \cdots \\
&\qquad + e^{-D(t_{n-i+h}-t_{n-i})} N_h(d_1(S, q_{i-1}, t_{n-i+1}-t_{n-i}), \ldots, d_1(S, q_{i-h+1}, t_{n-i+h+1}-t_{n-i}), \\
&\qquad - d_1(S, q_{i-h}, t_{n-i+h}-t_{n-i}); R^{(h)}) + \cdots \\
&\qquad + e^{-D(t_n-t_{n-i})} N_i(d_1(S, q_{i-1}, t_{n-i+1}-t_{n-i}), \ldots, d_1(S, q_1, t_{n-1}-t_{n-i}), \\
&\qquad - d_1(S, q_0, t_n-t_{n-i}); R^{(i)}), \\
&\qquad\quad i = 1, \ldots, n.
\end{aligned} \tag{3.3}
$$

The covariance matrix $R^{(h)}$ has entries:

$$\begin{aligned}\rho_{jk}^{(h)} &= \sqrt{\frac{t_{n-i+j}-t_{n-i}}{t_{n-i+k}-t_{n-i}}} \quad \text{if} \quad 1 \le j \le k \le h-1 \\ \rho_{jh}^{(h)} &= -\sqrt{\frac{t_{n-i+j}-t_{n-i}}{t_{n-i+h}-t_{n-i}}} \quad \text{if} \quad 1 \le j \le h.\end{aligned} \tag{3.4}$$

Now, the definitions of $q_0, \dots, q_{n-1}$ and $U_1(S), \dots, U_n(S)$, $W_1(S), \dots, W_n(S)$ are as follows. First we define $q_0 = X$ and

$$\begin{aligned}U_1(S) &= e^{-r(t_n-t_{n-1})}N_1(-d_2(S, q_0, t_n - t_{n-1})) \\ W_1(S) &= e^{-D(t_n-t_{n-1})}N_1(-d_1(S, q_0, t_n - t_{n-1})).\end{aligned}$$

Suppose that $U_1(S), \dots, U_i(S), W_1(S), \dots, W_i(S), q_1, \dots, q_{i-1}$ have already been defined; write up the equation:

$$X - S = XU_i(S) - SW_i(S) \tag{3.5}$$

and designate its unique solution (with respect to S) by q_i. Then we define $U_{i+1}(S)$ and $W_{i+1}(S)$ by (3.2) and (3.3), respectively. The existence and unity of q_i is a byproduct of the proof of the following

THEOREM 3.1 *We have the equations*

$$e^{-r\Delta t_{n-i+1}}E(V_{n-i+1}(Se^{-D\Delta t_{n-i+1}}Y_{n-i+1})) = XU_i(S) - SW_i(S), \qquad i = 1, \dots, n \tag{3.6}$$

and the price of the Bermudan put option is given by

$$P_n = \max(X - S, XU_n(S) - SW_n(S)). \tag{3.7}$$

PROOF. The latter assertion is a consequence of the former one. We prove the former assertion by induction on i. For $i = 1$ equation (3.6) is a special case of the Black–Scholes–Merton (BSM) formula (1.2)–(1.4) for the European put, if we use $t = t_{n-1}$, $T = t_n$.

Suppose that equation (3.6) is valid for all positive integers up to i. We prove that it is valid for $i + 1$ too.

If we use the induction hypothesis then we can write

$$V_{n-i}(S) = e^{-r\Delta t_{n-i}}E(\max(X - S, XU_i(S) - SW_i(S))).$$

It follows from this that

$$\begin{aligned}&e^{-r\Delta t_{n-i}}E(V_{n-i}(Se^{-D\Delta t_{n-i}}Y_{n-i})) \\ &= e^{-r\Delta t_{n-i}}\int_{-\infty}^{\infty}\max(X - Se^y, XU_i(Se^y) - Se^yW_i(Se^y)) \\ &\times\frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}}e^{-\frac{\left(y-\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2\Delta t_{n-i}}}\,\mathrm{dy}.\end{aligned} \tag{3.8}$$

We split the integral into two parts. In the first (second) part we integrate over the interval determined by $Se^y \leq q_i$ ($Se^y \geq q_i$). Thus we have the equality

$$
\begin{aligned}
& e^{-r\Delta t_{n-i}} E(V_{n-i}(SY_{n-i})) \\
&= e^{-r\Delta t_{n-i}} \int_{-\infty}^{\log \frac{q_i}{S}} (X - Se^y) \frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}}\, e^{-\frac{\left(y - \left(r - D - \frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2 \Delta t_{n-i}}} dy \\
&\quad + e^{-r\Delta t_{n-i}} \int_{\log \frac{q_i}{S}}^{\infty} (XU_i(Se^y) - Se^y W_i(Se^y)) \\
&\quad \times \frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}}\, e^{-\frac{\left(y - \left(r - D - \frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{\sigma^2 \Delta t_{n-i}}} dy. \qquad (3.9)
\end{aligned}
$$

The first term on the right hand side of (3.9) provides us with a BSM formula. This is the same as the first term in $XU_{i+1}(S) - SW_{i+1}(S)$, i.e., the term that arises in such a way that we plug in the values of $U_{i+1}(S)$ and $W_{i+1}(S)$ from (3.2) and (3.3) but then drop all terms where N_j, $j \geq 2$ appear. We show that the second integral on the right hand side of (3.9) is equal to the rest of $XU_{i+1}(S) - SW_{i+1}(S)$.

Let us split the above-mentioned second term into two terms, where

(a) $XU_i(Se^y)$ appears, and where

(b) $Se^y W_i(Se^y)$ appears.

First consider the term (b). The value of $W_i(S)$ is given by (3.3). Let us take from that sum the term that involves N_h. Then we are given the integral:

$$
\begin{aligned}
& e^{-r\Delta t_{n-i}} \int_{\log \frac{q_i}{S}}^{\infty} Se^y e^{-D(t_{n-i+h} - t_{n-i})} N_h \\
& \times \left(\frac{\log \frac{S}{q_{i-1}} + \left(r - D + \frac{\sigma^2}{2}\right)(t_{n-i+1} - t_{n-i}) + y}{\sigma\sqrt{t_{n-i+1} - t_{n-i}}}, \right. \\
& \frac{\log \frac{S}{q_{i-2}} + \left(r - D + \frac{\sigma^2}{2}\right)(t_{n-i+2} - t_{n-i}) + y}{\sigma\sqrt{t_{n-i+2} - t_{n-i}}}, \ldots, \\
& \frac{\log \frac{S}{q_{i-h+1}} + \left(r - D + \frac{\sigma^2}{2}\right)(t_{n-i+h-1} - t_{n-i}) + y}{\sigma\sqrt{t_{n-i+h-1} - t_{n-i}}},
\end{aligned}
$$

$$-\frac{\log\dfrac{S}{q_{i-h}}+\left(r-D+\dfrac{\sigma^2}{2}\right)(t_{n-i+h}-t_{n-i})+y}{\sigma\sqrt{t_{n-i+h}-t_{n-i}}};R^{(h)}\Bigg)$$

$$\times\frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}}\,e^{-\frac{\left(y-\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2\Delta t_{n-i}}}\,\mathrm{d}y. \tag{3.10}$$

Now we use the relation:

$$\begin{aligned}
&y-\frac{1}{2}\left(\frac{y-\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}}{\sigma\sqrt{\Delta t_{n-i}}}\right)^2-r\Delta t_{n-i}\\
&\quad=y-\frac{y^2-2\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}y+\left(r-D-\frac{\sigma^2}{2}\right)^2(\Delta t_{n-i})^2}{2\sigma^2\Delta t_{n-i}}\\
&\quad\quad-\frac{2\sigma^2 r-D(\Delta t_{n-i})^2}{2\sigma^2\Delta t_{n-i}}\\
&\quad=-\frac{\left(y-\left(r-D+\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2\Delta t_{n-i}}
\end{aligned}$$

and introduce the new variable

$$u=-\frac{y-\left(r-D+\frac{\sigma^2}{2}\right)\Delta t_{n-i}}{\sigma\sqrt{\Delta t_{n-i}}}.$$

Then (3.10) can be rewritten in the following form:

$$\begin{aligned}
&S\int_{-\infty}^{\left(\log\frac{S}{q_i}+\left(r-D+\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)/\sigma\sqrt{\Delta t_{n-i}}} e^{-D(t_{n-i+h}-t_{n-i})}N_h\\
&\quad\times\left(\frac{\log\frac{S}{q_{i-1}}+\left(r-D+\frac{\sigma^2}{2}\right)(t_{n-i+1}-t_{n-i-1})}{\sigma\sqrt{t_{n-i+1}-t_{n-i}}}-\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+1}-t_{n-i}}}u\right.\\
&\quad\frac{\log\frac{S}{q_{i-2}}+\left(r-D+\frac{\sigma^2}{2}\right)(t_{n-i+2}-t_{n-i-1})}{\sigma\sqrt{t_{n-i+2}-t_{n-i}}}-\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+2}-t_{n-i}}}u,\ldots,
\end{aligned}$$

$$
\frac{\log\dfrac{S}{q_{i-h+1}}+\left(r-D+\dfrac{\sigma^2}{2}\right)(t_{n-i+h-1}-t_{n-i-1})}{\sigma\sqrt{t_{n-i+h-1}-t_{n-i}}}-\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+h-1}-t_{n-i}}}u,
$$

$$
-\frac{\log\dfrac{S}{q_{i-h}}+\left(r-D+\dfrac{\sigma^2}{2}\right)(t_{n-i+h}-t_{n-i-1})}{\sigma\sqrt{t_{n-i+h}-t_{n-i}}}
$$

$$
+\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+h}-t_{n-i}}}u;R^{(h)}\Bigg)\,\varphi(u)\,\mathrm{d}u. \tag{3.11}
$$

At this point we make use of the formula (1.10) for the case of $n = h+1$,

$$
x_j=\frac{\log\dfrac{S}{q_j}+\left(r-D+\dfrac{\sigma^2}{2}\right)(t_{n-i+j-1}-t_{n-i-1})}{\sigma\sqrt{t_{n-i+j-1}-t_{n-i-1}}},\qquad j=1,\dots,h+1 \tag{3.12}
$$

and $R_0 = R^{(h)}$. By the inductive hypothesis the entries of $R^{(h)}$ are given by (3.4). What we have to show is that the integral (3.11) is equal to the term involving N_{h+1} in (3.2), written up for $i+1$.

In (3.12) we already specified the x_j, $j = 1,\dots,h+1$. Still we have to specify an $(h+1)\times(h+1)$ correlation matrix $R = (\rho_{jk})$ with which (1.10) holds true and then our task is to show that $\Phi(x_1,\dots,x_n;R)$ is equal to the above-mentioned term in the definition of $W_{i+1}(S)$. Let ρ_{1k}, $k = 1,\dots,h$, $\rho_{1,h+1}$ be defined by the equations

$$
\begin{aligned}
\frac{\rho_{1k}^2}{\sqrt{1-\rho_{1k}^2}}&=\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+k-1}-t_{n-i}}},\qquad k=2,\dots,h\\
\frac{\rho_{1,h+1}^2}{\sqrt{1-\rho_{1,h+1}^2}}&=-\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+h}-t_{n-i}}}.
\end{aligned} \tag{3.13}
$$

It follows that

$$
\begin{aligned}
\rho_{1k}&=\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+k-1}-t_{n-i-1}}},\qquad k=2,\dots,h\\
\rho_{1,h+1}&=-\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+h}-t_{n-i-1}}}.
\end{aligned} \tag{3.14}
$$

If we use these and take into account (1.9), then we can write up the equations for ρ_{jk}:

$$
\begin{aligned}
\rho_{jk}^{(h)}&=\sqrt{\frac{t_{n-i+j-1}-t_{n-i}}{t_{n-i+k-1}-t_{n-i}}}\\
&=\frac{\rho_{jk}-\rho_{1j}\rho_{1k}}{\sqrt{1-\rho_{1j}^2}\sqrt{1-\rho_{1k}^2}},\qquad 2\le j\le k\le h+1.
\end{aligned}
$$

From here and (3.14) we obtain

$$
\begin{aligned}
\rho_{jk} &= \sqrt{1-\rho_{1j}^2}\sqrt{1-\rho_{1k}^2}\rho_{jk}^{(h)} + \rho_{1j}\rho_{1k} \\
&= \sqrt{\frac{t_{n-i+j-1}-t_{n-i}}{t_{n-i+k-1}-t_{n-i}}}\sqrt{\frac{t_{n-i+j-1}-t_{n-i}}{t_{n-i+j-1}-t_{n-i-1}}}\sqrt{\frac{t_{n-i+j-1}-t_{n-i}}{t_{n-i+k-1}-t_{n-i}}} \\
&\quad + \sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+k-1}-t_{n-i-1}}}\sqrt{\frac{t_{n-i}-t_{n-i-1}}{t_{n-i+j-1}-t_{n-i-1}}} \\
&= \sqrt{\frac{t_{n-i+j-1}-t_{n-i-1}}{t_{n-i+k-1}-t_{n-i-1}}}, \qquad 2 \le j \le k \le h. \qquad (3.15)
\end{aligned}
$$

Similarly, we obtain

$$
\rho_{j,h+1} = -\sqrt{\frac{t_{n-i+j-1}-t_{n-i-1}}{t_{n-i+h}-t_{n-i}}}. \qquad (3.16)
$$

From (3.14), (3.15) and (3.16) we can collect all entries of R and we can see that $R = R^{(h+1)}$.

Going back to equations (3.6), what we have proved so far is that if we split both integrals on the right hand side into two terms, by taking the terms with positive and negative signs separately, then the sum of the negative terms, multiplied by -1, is equal to $SW_{i+1}(S)$, i.e.,

$$
\begin{aligned}
& e^{-r\Delta t_{n-i}}\int_{-\infty}^{\log\frac{q_i}{S}} Se^y \frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}}\, e^{-\frac{\left(y-\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2\Delta t_{n-i}}}\mathrm{d}y \\
& +e^{-r\Delta t_{n-i}}\int_{\log\frac{q_i}{S}}^{\infty} Se^y W_i(Se^y)\frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}} e^{-\frac{\left(y-\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2\Delta t_{n-i}}}\mathrm{d}y \\
&= SW_{i+1}(S). \qquad (3.17)
\end{aligned}
$$

In the same way we can prove that

$$
\begin{aligned}
& e^{-r\Delta t_{n-i}}\int_{-\infty}^{\log\frac{q_i}{S}} X \frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}}\, e^{-\frac{\left(y-\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2\Delta t_{n-i}}}\mathrm{d}y \\
& +e^{-r\Delta t_{n-i}}\int_{\log\frac{q_i}{S}}^{\infty} XU_i(Se^y)\frac{1}{\sqrt{2\pi}\sigma\sqrt{\Delta t_{n-i}}} e^{-\frac{\left(y-\left(r-D-\frac{\sigma^2}{2}\right)\Delta t_{n-i}\right)^2}{2\sigma^2\Delta t_{n-i}}}\mathrm{d}y \\
&= XU_{i+1}(S). \qquad (3.18)
\end{aligned}
$$

Thus, the equation in Theorem 3.1 holds true for $i+1$, too. This proves the theorem. $\square$

4 Formulas for the Price of the Bermudan call option

In this section we use the notations (3.1) unchanged. We also use the notations $V_0(S)$, $\dots, V_n(S)$, $U_1(S), \dots, U_n(S)$, $W_1(S), \dots, W_n(S)$ and $q_0, q_1, \dots, q_{n-1}$ but they have different meaning than in Section 3, except for q_0 which remains equal to X.

The dynamic programming recursions can be obtained from (4.1) by replacing $[S-X]_+$ for $[X-S]_+$. Thus, the equations for the Bermudan call with dividend, paid continuously in time at rate D, are

$$\begin{aligned} &V_n(S) = [S-X]_+ \\ &V_i(S) = \max\left([S-X]_+, e^{-r\Delta t_{i+1}} E\left(V_{i+1}(Se^{-D\Delta t_{i+1}} Y_{i+1})\right)\right) \\ &i = 1, \dots, n-1. \end{aligned} \tag{4.1}$$

The following three theorems are counterparts of those in Section 2 and their proofs are the same.

THEOREM 4.1 *The following assertions hold true*

(a) $V_i(0) = 0$, $i = 0, \dots, n$,

(b) *for every* $i = 0, \dots, n-1$ *the function* $V_i(S)$ *is continuous and strictly increasing for* $S \geq 0$.

THEOREM 4.2 *Suppose that* $\Delta t_i = \Delta t = (T-t)/n$, $i = 1, \dots, n$. *Then for any* $S \geq 0$ *we have the relation*

$$V_{i-1}(S) \geq V_i(S).$$

The inequality is strict if $S > q_{n-i+1}$ *and equality holds if* $S \leq q_{n-i+1}$. *In the latter case* $V_{i-1}(S) = V_i(S) = X - S$. *In addition, we have the inequalities* (4.5).

THEOREM 4.3 *If* τ_1, τ_2 *are two equidistant subdivisions and* $\tau_1 \prec \tau_2$, *then, the value of the Bermudan option corresponding to* τ_1 *is not greater than the value corresponding to* τ_2.

Let us define $U_i(S)$ and $W_i(S)$ in the following way:

$$\begin{aligned} &U_i(S) \\ &\quad = e^{-D(t_{n-i+1}-t_{n-i})} N_1(d_1(S, q_{i-1}, t_{n-i+1} - t_{n-i})) \\ &\quad\quad + e^{-D(t_{n-i+2}-t_{n-i})} N_2(-d_1(S, q_{i-1}, t_{n-i+1} - t_{n-i}), d_1(S, q_{i-2}, t_{n-i+2} - t_{n-i}); R^{(2)}) \\ &\quad\quad + e^{-D(t_{n-i+3}-t_{n-i})} N_3(-d_1(S, q_{i-1}, t_{n-i+1} - t_{n-i}), \end{aligned}$$

$$-d_1(S_1, q_{i-2}, t_{n-i+2} - t_{n-i}), d_1(S, q_{i-3}, t_{n-i+3} - t_{n-i}); R^{(3)}) + \cdots$$

$$+e^{-D(t_{n-i+h}-t_{n-i})} N_h(-d_1(S, q_{i-1}, t_{n-i+1} - t_{n-i}), \ldots,$$

$$-d_1(S, q_{i-h+1}, t_{n-i+h-1} - t_{n-i}), d_1(S, q_{i-h}, t_{n-i+h} - t_{n-h}); R^{(h)}), + \cdots$$

$$+e^{-D(t_n-t_{n-i})} N_i(-d_1(S, q_{i-1}, t_{n-i+1} - t_{n-i}), \ldots,$$

$$-d_1(S, q_1, t_{n-1} - t_{n-i}), d_1(S, q_0, t_n - t_{n-i}) : R^{(i)}),$$

$$i = 1, \ldots, n, \tag{4.2}$$

$$W_i(S)$$

$$= e^{-r(t_{n-i+1}-t_{n-i})} N_1(d_2(S, q_{i-1}, t_{n-i+1} - t_{n-i}))$$

$$+e^{-r(t_{n-i+2}-t_+n-i)} N_2(-d_2(S, q_{i-1}, t_{n-i+1} - t_{n-i}), d_2(S, q_{i-2}, t_{n-i+2} - t_{n-i}); R^{(2)})$$

$$+e^{-r(t_{n-i+3}-t_{n-i})} N_3(-d_2(S, q_{i-1}, t_{n-i+1} - t_{n-i}),$$

$$-d_2(S, q_{i-2}, t_{n-i+2} - t_{n-i}), d_2(S, q_{i-3}, t_{n-i+3} - t_{n-i}); R^{(3)}) + \cdots$$

$$+e^{-r(t_{n-i+h}-t_{n-i})} N_h(-d_2(S, q_{i-1}, t_{n-i+1} - t_{n-i}), \ldots,$$

$$-d_2(S, q_{i-h+1}, t_{n-i+h+1} - t_{n-i}), d_2(S, q_{i-h}, t_{n-i+h} - t_{n-i}); R^{(h)}) + \cdots$$

$$+e^{-r(t_n-t_{n-i})} N_i(-d_2(S, q_{i-1}, t_{n-i+1} - t_{n-i}), \ldots,$$

$$-d_2(S, q_1, t_{n-1} - t_{n-i}), d_2(S, q_0, t_n - t_{n-i}); R^{(i)}),$$

$$i = 1, \ldots, n. \tag{4.3}$$

We define $q_1, \ldots, q_{n-1}$ recursively in a similar way as we have defined the corresponding values in connection with the Bermudan put. The equation for q_i in case of the Bermudan call is:

$$S - X = e^{-D\Delta t_{n-i+1}}(SU_i(S) - XW_i(S)). \tag{4.4}$$

Having q_i we write up the formulas for $U_{i+1}(S)$, $W_{i+1}(S)$ and then proceed the same way. For the obtained values we have the inequalities

$$q_0 = X < q_1 < \cdots < q_{n-1} \tag{4.5}$$

The next theorem is the counterpart of Theorem 3.1.

Theorem 4.4 *We have the equations*

$$e^{-D\Delta t_{n-i+1}} E(V_{n-i+1}(Se^{-D\Delta t_{n-i+1}} Y_{n-i+1})) = SU_i(S) - XW_i(S), \qquad i = 1, \ldots, n$$

and the value of the Bermudan call option is:

$$C_n = \max(S - X, SU_n(S) - XW_n(S)). \tag{4.6}$$

If there is no dividend and C designates the price of the American call option, then, as it is well-known, $C = c$.

5 The American options

For any positive integer n we define the function $q^{(n)}(\tau)$, $t \le \tau \le T$ in the following way. Let $t = t_0 < t_1 < \cdots < t_n = T$ be a subdivision of the interval $[t, T]$ and compute $q_1, \ldots, q_{n-1}$ defined in Section 3. Then let $q_0 = X$, as before, $q_n - q_{n-1}$ and

$$q^{(n)}(\tau) = \frac{q_{i-1} - q_i}{t_i - t_{i-1}}(\tau - t_{i-1}), \qquad i = 1, \ldots, n.$$

We have the following

THEOREM 5.1 *If* $\max_{1 \le i \le n} \Delta t_i \to 0$ *as* $n \to \infty$, *then for any* $\tau \in [t, T]$ $q^{(n)}(\tau)$ *is convergent and the function*

$$q(\tau) = \lim_{n \to \infty} q^{(n)}(\tau), \qquad t \le \tau \le T \tag{5.1}$$

is continuous, strictly decreasing and convex.

PROOF. Omitted. □

Before stating the next theorem we mention that $U_i(S)$ and $W_i(S)$ have simple probabilistic meaning. In fact, for $U_n(S)$ we can write

$$\begin{aligned} U_n(S) & \\ &= e^{-r(t_1-t_0)} P\left(Se^{\sigma(B(t_1)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t_1-t_0)} > q_{n-1} \right) \\ &\quad + \sum_{h=2}^{n} e^{-r(t_h-t_0)} P\left(Se^{\sigma(B(t_l)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t_l-t_0)} < q_{n-l}, \right. \\ &\quad \left. l = 1, \ldots, h-1,\ Se^{\sigma(B(t_h)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t_h-t_0)} > q_{n-h} \right) \end{aligned}$$

$= P$ (present value of \$ 1.00 discounted at the constant rate r, paid upon the time the geometric Brownian motion

$$X(t) = Se^{\sigma(B(t)-B(t_0))+\left(r-D-\frac{\sigma^2}{2}\right)(t-t_0)}$$

first reaches or exceeds the discrete critical function q_{n-h}, $h = 1, \ldots, n$, at some point t_h, $h = 1, \ldots, n$), (5.2) where $B(t)$, $t \geq 0$ is a standard Brownian motion process. Similarly, we have that

$$\begin{aligned}
&W_n(S) \\
&= e^{-D(t_1-t_0)} P\left(S e^{\sigma(B(t_1)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t_1-t_0)} > q_{n-1} \right) \\
&\quad + \sum_{h=2}^{n} e^{-D(t_h-t_0)} P\Bigg(S e^{\sigma(B(t_l)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t_l-t_0)} < q_{n-l}, \\
&\qquad l = 1, \ldots, h-1,\ S e^{\sigma(B(t_h)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t_h-t_0)} > q_{n-h} \Bigg)
\end{aligned}$$

$= P$ (present value of \$ 1.00, discounted at the constant rate D, paid upon the time the geometric Brownian motion

$$X(t) = S e^{\sigma(B(t)-B(t_0))+\left(r-D+\frac{\sigma^2}{2}\right)(t-t_0)}$$

first reaches or exceeds the discrete critical function q_{n-h}, $h = 1, \ldots, n$, at some point t_h, $h = 1, \ldots, n$), (5.3)

where $B(t)$, $t \geq 0$ is a standard Brownian motion process.

The correlation matrix of the random variables $B(t_1) - B(t_0), \ldots, B(t_n) - B(t_0)$ is:

$$\begin{pmatrix}
\sqrt{\frac{t_1-t_0}{t_1-t_0}} & \sqrt{\frac{t_1-t_0}{t_2-t_0}} & \sqrt{\frac{t_1-t_0}{t_3-t_0}} & \cdots & \sqrt{\frac{t_1-t_0}{t_n-t_0}} \\
\sqrt{\frac{t_1-t_0}{t_2-t_0}} & \sqrt{\frac{t_2-t_0}{t_2-t_0}} & \sqrt{\frac{t_2-t_0}{t_3-t_0}} & \cdots & \sqrt{\frac{t_2-t_0}{t_n-t_0}} \\
\sqrt{\frac{t_1-t_0}{t_3-t_0}} & \sqrt{\frac{t_2-t_0}{t_3-t_0}} & \sqrt{\frac{t_3-t_0}{t_3-t_0}} & \cdots & \sqrt{\frac{t_3-t_0}{t_n-t_0}} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\sqrt{\frac{t_1-t_0}{t_n-t_0}} & \sqrt{\frac{t_2-t_0}{t_n-t_0}} & \sqrt{\frac{t_3-t_0}{t_n-t_0}} & \cdots & \sqrt{\frac{t_n-t_0}{t_n-t_0}}
\end{pmatrix}. \tag{5.4}$$

The matrix R_h in formulas (3.2) and (3.3) can be obtained from the matrix (5.4). In fact, if we take the matrix traced out from (5.4) by the first h rows and columns and then multiply by -1 all offdiagonal entries in the hth row and hth column, then we obtain R_h.

The probabilistic interpretation of $U_n(S)$ and $W_n(S)$ imply that the limits

$$U(S) = \lim_{n\to\infty} U_n(S) \tag{5.5}$$

$$W(S) = \lim_{n\to\infty} W_n(S) \tag{5.6}$$

exist. In fact, any Brownian motion process has a representation, where almost all sample functions are continuous (see, e.g., Doob, 1953) which imply the existence of the limits (5.5), (5.6).

The limiting function $W(S)$ equals the probability that the multiplicative Brownian motion in (5.3) intersects the critical price function $q(\tau)$, $t \leq \tau \leq T$. The other limiting function equals the present value of \$ 1.00 paid upon the time the multiplicative Brownian motion in (5.2) intersects $q(\tau)$, $t \leq \tau \leq T$.

The price of the American put option is

$$P = \max(X - S, XU(S) - SW(S)). \tag{5.7}$$

Finally, the price of the American call with dividend is equal to the following

$$C = \max(S - X, SU(S) - XW(S)). \tag{5.8}$$

6 On the Binomial Tree Method

In this section we show that under some conditions, frequently satisfied in the numerical calculation, the binomial tree method systematically overestimates the value of any option of which the payoff is a convex function of the spot price S. This is a simple consequence of a theorem known in the theory of the univariate moment problem.

Let X be a random variable the support of which is the finite interval $I = [a, b]$. Suppose that I, $E(X) = \mu$ are known but the c.d.f. $F(z)$ of X is unknown. Let $f(z)$, $a \leq z \leq b$ be a nonlinear convex function. Then the optimal solution of the problem:

$$\begin{aligned}
&\max \int_I f(z)\,\mathrm{d}F(z) \\
&\text{subject to} \\
&\int_I z\,\mathrm{d}F(z) = \mu
\end{aligned} \tag{6.1}$$

provides us with a probability distribution with support $\{a, b\}$ (see, e.g., Karlin and Studden, 1966). The corresponding probabilities can be determined from the equations

$$\begin{aligned}
ap + bq &= \mu \\
q &= 1 - p
\end{aligned}$$

and the result is

$$p = \frac{b - \mu}{b - a}, \qquad q = \frac{\mu - a}{b - a}. \tag{6.2}$$

This implies the relation

$$E[f(X)] \leq f(a)\frac{b - \mu}{b - a} + f(b)\frac{\mu - a}{b - a}, \tag{6.3}$$

known also as the Edmundson–Madansky inequality (see, e.g. Prékopa, 1995). It provides us with a sharp upper bound for the expectation of $f(X)$. In (6.3) equality holds if and only if the support of X is the set $\{a, b\}$.

We can apply this result for the calculation of an option value using the binomial tree method. Consider a node, labeled 0, and its two descendants, nodes 1 and 2. Suppose that at node 0 the spot price is S_0 and at the descendants the prices $S_0 d$ and $S_0 u$ are assigned, respectively. If $f(S)$ is the payoff function, then the binomial tree method assigns the option price

$$e^{-r\Delta t}(pf(S_0 d) + qf(S_0 u)), \qquad q = 1 - p \tag{6.4}$$

to node 0. The probabilities p, q can be determined from the no arbitrage equation, that we assume to hold:

$$e^{-r\Delta t}(p(S_0 d) + q(S_0 u)) = S_0, \qquad q = 1 - p.$$

The result is:

$$p = \frac{e^{r\Delta t} - d}{u - d}, \qquad q = \frac{u - e^{r\Delta t}}{u - d}. \tag{6.5}$$

Note that the values d, u have to be chosen in such a way that $d < e^{r\Delta t} < u$.

Now, assume that S_0 corresponds to time 0 and at time 1 the price of the asset is a random variable S for which we have

$$P(S_0 d \le S \le S_0 u) = 1. \tag{6.6}$$

If $F(s)$ is the c.d.f. of S, then the true option price at time 0 is

$$e^{-r\Delta t} \int_I f(s)\,\mathrm{d}F(s) \tag{6.7}$$

that is always smaller than or equal to the value in (6.4), with p, q given by (6.5). In fact, this is a consequence of the inequality (6.3). In practice the possible values of the random price S is a larger set, then the set of endpoints of the interval $[S_0 d, S_0 u]$ and condition (6.6) is frequently satisfied. It follows that the true option price (6.7) is typically strictly smaller than the value in (6.4).

Summarizing: if equation (6.6) holds true for any node in the binomial tree, the no arbitrage assumption holds at each node and the option has a convex payoff, then the binomial tree method always provides us with an upper bound for the option value. In practice it translates into an overestimation of the option value. The overestimation disappears in the limit, when $\Delta t \to 0$ and d, u are suitably chosen in the limiting procedure (see Cox, Rubinstein, 1985). However, since the difference between the standardized binomial c.d.f. with parameter n and the standard normal c.d.f. is of order of magnitude $1/\sqrt{n}$, by the Berry–Esséen theorem (see, e.g., Feller, 1972), the choice $n = 10,000$ gives only two digit accuracy.

Note that a typical choice of u and d is $u = 1.2$, $d = 1/1.2$. In this case the condition (6.6) means that the change of the price between times 0 and 1 is not greater than 20% in both the upward and downward directions.

7 On the Richardson Approximation

Richardson extrapolation designates a collection of methods, where we determine the limiting value of an analytic function of a stepsize z where the latter goes to zero. The values of the analytic function are known exactly or approximately for a finite number of h values and the fitting of the function is part of the procedure.

The Richardson extrapolation we are using here first determines the Lagrange polynomial $L(z)$ that takes given values $f_1, \dots, f_n$ at given base points $z_1, \dots, z_n$, respectively, and then extrapolates the function value at $z = 0$, by the use of $L(0)$. We apply it for two different cases.

Case I. Suppose that $h > 0$, the base points are:

$$\frac{h}{n}, \frac{h}{n-1}, \dots, \frac{h}{2}, h \tag{7.1}$$

and the corresponding function values are:

$$P_n, P_{n-1}, \dots, P_2, P_1, \tag{7.2}$$

or

$$C_n, C_{n-1}, \dots, C_2, C_1. \tag{7.3}$$

It is easy to see that $L(0)$ does not change if each base point is multiplied by the same positive number, hence we may choose $h = 1$.

Another way to formulate the method is the following. Solve for $a_0, a_1, \dots, a_n$ the equations

$$P_i = \sum_{k=0}^{n} a_k \frac{1}{i^k}, \qquad i = n, n-1, \dots, 1 \tag{7.4}$$

if we want to approximate the American put, or the similar equations, written up with C_i on the left hand side, if we want to approximate the American call. Then the Lagrange polynomial is

$$L(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

and $L(0) = a_0$ is the extrapolating value. Since the Lagrange polynomial has also the form:

$$L(z) = \sum_{i=1}^{n} f_i \frac{(z - z_1) \cdots (z - z_{i-1})(z - z_{i+1}) \cdots (z - z_n)}{(z_i - z_1) \cdots (z_i - z_{i-1})(z_i - z_{i+1}) \cdots (z_i - z_n)} \tag{7.5}$$

with the current base points (7.1), then a simple calculation shows that in case of the function values (7.2), (7.3) the following closed form formulas provide us with the values $L(0) = P(1 \dots n)$, $L(0) = C(1 \dots n)$:

$$P(1 \dots n) = \sum_{j=1}^{n} (-1)^{n-j} \frac{j^{n-1}}{(n-j)!(j-1)!} P_j, \tag{7.6}$$

$$C(1\dots n)=\sum_{j=1}^{n}(-1)^{n-j}\frac{j^{n-1}}{(n-j)!(j-1)!}C_j. \tag{7.7}$$

Case II. In the second case our base points are

$$2^{-(n-1)},2^{-(n-2)},\dots,2^{-1},2^0=1 \tag{7.8}$$

and the corresponding function values are

$$P_{2^{n-1}},P_{2^{n-2}},\dots,P_2,P_1, \tag{7.9}$$

or

$$C_{2^{n-1}},C_{2^{n-2}},\dots,C_2,C_1. \tag{7.10}$$

If we plug the base points (7.8) into the Lagrange polynomial (7.5), then a simple calculation shows that, designating by $P(12\dots 2^{n-1})$ and $C(12\dots 2^{n-1})$ the value $L(0)$, depending on if we use (7.9) or (7.10) as function values, respectively, we have the equations:

$$P(12\dots 2^{n-1})=\sum_{i=1}^{n}(-1)^{n-i}\frac{2^{\frac{(2i-n)(n-1)}{2}}}{\prod_{j=1}^{n-i}(2^j-1)\prod_{j=1}^{i-1}(2^j-1)}P_{2^{i-1}}, \tag{7.11}$$

$$C(12\dots 2^{n-1})=\sum_{i=1}^{n}(-1)^{n-i}\frac{2^{\frac{(2i-n)(n-1)}{2}}}{\prod_{j=1}^{n-i}(2^j-1)\prod_{j=1}^{i-1}(2^j-1)}C_{2^{i-1}}. \tag{7.12}$$

In the tables presented in the next section the numerical values of $P(1\dots n)$, $C(1\dots n)$ are shown for $n=4,\dots,8$ and the numerical values of $P(12\dots 2^{n-1})$, $C(12\dots 2^{n-1})$ are shown for $n=3,4$. Geske and Johnson (1984) calculated $P(1\dots 4)$. Even though our Richardson extrapolation is more powerful than the earlier one, applied in the same context ($n=4$), our final proposal to approximate the American option values is different. We propose an exponential smoothing of the sequences $P_1,\dots P_n$ and $C_1,\dots C_n$, respectively and then take the limiting values of the obtained discrete exponential functions, as $n\to\infty$, as approximations of the values of the American options.

8 Brief Description of the Applied Integration and Simulation Techniques. Numerical Results.

In order to obtain the numerical values of the Bermudan put and call options, we need numerical integration technique that provides us with the values of the normal c.d.f. in higher dimensions. With the dimension n we go up to the point where the calculation of the Bermudan option values P_n, C_n is still reliable.

Based on extensive experience with the integration of the multivariate normal p.d.f., we have chosen three methods to apply here. One is due to Genz (1992, 1996), the other one is due to Szántai (2000) and the third one is due to Ambartzumian et al. (1998).

The second one is a collection of methods and works in such a way that we take lower and upper bounds for the probability of a union of events and then use simulation for the difference. If A_i is the event $\{X_i > x_i\}$, $i = 1, \ldots, n$ and we want to estimate $P(X_i \le x_i,\ i = 1, \ldots, n)$, then first we estimate the probability of the union $\bigcup_{i=1}^n A_i$ and then estimate

$$P(X_i \le x_i,\ i = 1, \ldots, n) = 1 - P(A_1 \cup \ldots \cup A_n).$$

For the probability of the union prominent bounds, based on binomial moments (the $S_1, S_2, \ldots$ that appear in the inclusion-exclusion formula), and graph structures are available. The binomial moment bounds that are used in this context are those, presented in Boros and Prékopa (1989), for the cases, where only S_1, S_2, S_3, S_4 are used and in Prékopa (1988), were $S_1, \ldots, S_m,\ m > 4$ are used. The graph structure bounds utilized here are those of Bukszár and Prékopa (2001), Bukszár and Szántai (2002) and Bukszár (2002).

When the subdivision of the interval $[t, T]$ is equidistant, the multivariate normal random vectors can be generated in a much faster way than in the general case. In this case the correlation matrix given by (5.4) takes the form

$$R = \begin{pmatrix} 1 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & \cdots & \sqrt{\frac{1}{n}} \\ \sqrt{\frac{1}{2}} & 1 & \sqrt{\frac{2}{3}} & \cdots & \sqrt{\frac{2}{n}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 1 & \cdots & \sqrt{\frac{3}{n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{1}{n}} & \sqrt{\frac{2}{n}} & \sqrt{\frac{3}{n}} & \cdots & 1 \end{pmatrix}, \tag{8.1}$$

and its Cholesky factor is

$$T = \begin{pmatrix} 1 & & & & \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & & & \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \sqrt{\frac{1}{n}} & \sqrt{\frac{1}{n}} & \sqrt{\frac{1}{n}} & \cdots & \sqrt{\frac{1}{n}} \end{pmatrix}. \tag{8.2}$$

If we multiply the ith argument of the c.d.f. that we estimate, by $\sqrt{i}, i = 1, \ldots, n$, then in order to generate the required random numbers we can simply take the partial sums of the independent, standard normal random numbers. There is no multiplication in the procedure.

After realizing that the calculated multivariate normal probabilities are frequently very small, the sequential conditioned importance sampling (SCIS) procedure by Ambartzumian et al. (1998) was also tested.

All the three simulation and numerical integration procedures produced essentially the same results. As regards accuracy, the individual integrals have been computed with 5 digit precision on a confidence level of 99%. We claim 3 digit accuracy in the values of the Bermudan options.

In Tables 2,3,4,5 the "True" values are those obtained by the authors of the mentioned papers, by the use of the binomial tree method using large numbers of steps.

In the calculation of $P_1, \ldots, P_n$ and $C_1, \ldots, C_n$ it is unavoidable that some numerical errors occur. To overcome this difficulty we use of the following discrete exponential function

$$f(n) = k - \sum_{i=1}^{m} \alpha_i e^{-\beta_i n} \tag{8.3}$$

to smooth the above sequences. The constants $k, \alpha_i, \beta_i, i = 1, \ldots, m$ are determined by the least squares principle:

$$\min_{\substack{k,\alpha_1,\ldots,\alpha_m,\\ \beta_1,\ldots,\beta_m}} \sum_{i=1}^{n} (f(i) - P_i)^2. \tag{8.4}$$

In our examples it was enough to choose $m = 1$ or $m = 2$, to obtain very good fit.

The use of the type of function (8.3) is supported by the fact that all jth order differences of the sequences $\{P_i\}$, $\{C_i\}$ are of the same sign and the signs are alternating as j varies. The same property is enjoyed by the function (8.3).

In the tables that follow we compare our numerical results to those in Geske and Johnson (1984), Broadie and Detemple (1996), Arciniega and Allen (2004), Ju (1998) and create new numerical examples. Our figures, obtained by the exponential smoothing procedure, are smaller than those, obtained by others, for the same input data and significantly smaller than tha "True" value. The latters are obtained by the binomial tree calculation, using $10,000 - 15,000$ steps. Earlier we have remarked that $n = 10,000$ is not enough to use, and so is $n = 15,000$, as number of steps to obtain accurate result. In fact, the effect, discussed in Section 6, seems to be working and producing overestimation.

The closest to our figures are those obtained by Geske and Johnson (1984) by the use of Richardson extrapolation with P_1, P_2, P_3, P_4. However, it is shown in Tables 1, 2, 3, 4 that the use of Richardson extrapolation, applied for the cases $P_1, \ldots, P_n$, $5 \leq n \leq 8$ does not stabilize the extrapolation and as our numerical experience shows the situation is even worse if we use $n = 9, \ldots, 20$. This tells us that the Richardson extrapolation is an unreliable procedure, at least in this context.

In Tables 10 and 12 we present numerical evidence that the subsequent differences of the sequences of option prices $C_1, \ldots, C_n$ and $P_1, \ldots, P_n$ have the alternating sign property, at least to a high degree. Violations of this property occur in case of very small numbers.

Table 11 corresponds to the same parameters as Table 10. It shows the critical stock prices for the Bermudan calls: exercise takes place whenever the spot payoff is greater than or equal to the current critical price, for the first time.

Table 1-a: Geske-Johnson problems, I.

	Problem No.									
	1	2	3	4	5	6	7	8	9	10
S	1	1	1	1	1	1	1	1	1	1
r	0.1250	0.0800	0.0450	0.0200	0.0050	0.0900	0.0400	0.0100	0.0800	0.0200
X	1	1	1	1	1	1	1	1	1	1
σ	0.5	0.4	0.3	0.2	0.1	0.3	0.2	0.1	0.2	0.1
T	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
P_1	0.1327	0.1170	0.0959	0.0694	0.0373	0.0761	0.0600	0.0349	0.0442	0.0304
P_2	0.1408	0.1216	0.0981	0.0701	0.0374	0.0814	0.0620	0.0352	0.0487	0.0313
P_3	0.1431	0.1229	0.0988	0.0704	0.0375	0.0830	0.0626	0.0354	0.0501	0.0316
P_4	0.1443	0.1236	0.0991	0.0705	0.0375	0.0837	0.0629	0.0354	0.0507	0.0317
P_5	0.1449	0.1241	0.0994	0.0706	0.0376	0.0842	0.0631	0.0355	0.0511	0.0318
P_6	0.1454	0.1244	0.0996	0.0707	0.0376	0.0845	0.0633	0.0355	0.0514	0.0319
P_7	0.1458	0.1246	0.0997	0.0708	0.0376	0.0847	0.0634	0.0355	0.0516	0.0319
P_8	0.1460	0.1248	0.0998	0.0708	0.0376	0.0849	0.0635	0.0356	0.0517	0.0320
P_9	0.1462	0.1249	0.0999	0.0708	0.0376	0.0850	0.0635	0.0356	0.0518	0.0320
P_{10}	0.1464	0.1250	0.0999	0.0709	0.0376	0.0851	0.0636	0.0356	0.0519	0.0320
P_{11}	0.1465	0.1251	0.1000	0.0709	0.0376	0.0852	0.0636	0.0356	0.0520	0.0320
P_{12}	0.1467	0.1251	0.1000	0.0709	0.0377	0.0853	0.0637	0.0356	0.0520	0.0321
P_{13}	0.1467	0.1253	0.1000	0.0709	0.0376	0.0854	0.0637	0.0356	0.0521	0.0321
P_{14}	0.1468	0.1253	0.1000	0.0709	0.0376	0.0854	0.0637	0.0357	0.0521	0.0321
P_{15}	0.1469	0.1254	0.1001	0.0709	0.0376	0.0854	0.0637	0.0356	0.0522	0.0321
P_{16}	0.1470	0.1254	0.1001	0.0710	0.0376	0.0855	0.0637	0.0357	0.0522	0.0321
P_{17}	0.1471	0.1254	0.1001	0.0710	0.0376	0.0855	0.0637	0.0357	0.0522	0.0321
P_{18}	0.1471	0.1254	0.1002	0.0709	0.0377	0.0856	0.0638	0.0356	0.0523	0.0321
P_{19}	0.1471	0.1255	0.1002	0.0710	0.0377	0.0856	0.0638	0.0356	0.0523	0.0321
P_{20}	0.1472	0.1255	0.1002	0.0710	0.0377	0.0856	0.0638	0.0357	0.0523	0.0322
Exp. smoothed	0.1468	0.1255	0.1002	0.0710	0.0377	0.0857	0.0638	0.0357	0.0524	0.0322
G–J	0.1476	0.1258	0.1005	0.0712	0.0377	0.0859	0.0640	0.0357	0.0525	0.0322
$P(1\ldots4)$	0.1476	0.1259	0.1005	0.0711	0.0377	0.0859	0.0640	0.0357	0.0525	0.0322
$P(1\ldots5)$	0.1481	0.1261	0.1006	0.0711	0.0377	0.0861	0.0641	0.0357	0.0527	0.0323
$P(1\ldots6)$	0.1480	0.1260	0.1003	0.0710	0.0378	0.0863	0.0641	0.0358	0.0527	0.0322
$P(1\ldots7)$	0.1481	0.1263	0.1012	0.0714	0.0373	0.0857	0.0639	0.0353	0.0534	0.0324
$P(1\ldots8)$	0.1488	0.1250	0.0966	0.0694	0.0390	0.0882	0.0656	0.0375	0.0506	0.0314
$P(124)$	0.1473	0.1255	0.1002	0.0711	0.0377	0.0858	0.0638	0.0357	0.0526	0.0321
$P(1248)$	0.1479	0.1260	0.1005	0.0711	0.0377	0.0861	0.0641	0.0357	0.0526	0.0323

Table 1-b: Geske-Johnson problems, II.

	Problem No.									
	11	12	13	14	15	16	17	18	19	20
S	1	1	40	40	40	40	40	40	40	40
r	0.1200	0.0300	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490
X	1	1	35	35	35	40	40	40	45	45
σ	0.2	0.1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
T	1.0000	1.0000	0.0833	0.3333	0.5833	0.0833	0.3333	0.5833	0.0833	0.3333
P_1	0.0317	0.0263	0.0062	0.1960	0.4170	0.8403	1.5221	1.8812	5.0000	5.0000
P_2	0.0381	0.0278	0.0062	0.1964	0.4200	0.8443	1.5478	1.9339	5.0000	5.0000
P_3	0.0402	0.0283	0.0062	0.1972	0.4232	0.8465	1.5569	1.9506	5.0000	5.0077
P_4	0.0413	0.0285	0.0063	0.1978	0.4250	0.8477	1.5620	1.9595	5.0000	5.0322
P_5	0.0418	0.0286	0.0061	0.1982	0.4263	0.8485	1.5653	1.9652	5.0000	5.0458
P_6	0.0422	0.0287	0.0061	0.1985	0.4272	0.8490	1.5676	1.9691	5.0000	5.0545
P_7	0.0425	0.0288	0.0062	0.1986	0.4279	0.8495	1.5692	1.9720	5.0000	5.0604
P_8	0.0427	0.0289	0.0062	0.1988	0.4285	0.8498	1.5704	1.9742	5.0000	5.0646
P_9	0.0428	0.0289	0.0061	0.1990	0.4288	0.8500	1.5721	1.9766	5.0000	5.0680
P_{10}	0.0430	0.0289	0.0063	0.1990	0.4296	0.8503	1.5720	1.9778	5.0000	5.0704
P_{11}	0.0431	0.0290	0.0061	0.1991	0.4298	0.8499	1.5731	1.9794	5.0000	5.0728
P_{12}	0.0431	0.0290	0.0062	0.1994	0.4291	0.8502	1.5741	1.9802	5.0000	5.0739
P_{13}	0.0432	0.0290	0.0062	0.1992	0.4304	0.8509	1.5739	1.9809	5.0000	5.0755
P_{14}	0.0433	0.0290	0.0062	0.1990	0.4304	0.8515	1.5748	1.9805	5.0000	5.0756
P_{15}	0.0433	0.0291	0.0061	0.1996	0.4302	0.8508	1.5751	1.9810	5.0000	5.0763
P_{16}	0.0434	0.0291	0.0060	0.1999	0.4304	0.8502	1.5735	1.9834	5.0000	5.0785
P_{17}	0.0434	0.0290	0.0063	0.1990	0.4294	0.8508	1.5760	1.9843	5.0000	5.0789
P_{18}	0.0434	0.0291	0.0063	0.1995	0.4292	0.8515	1.5755	1.9840	5.0000	5.0774
P_{19}	0.0435	0.0291	0.0058	0.1996	0.4316	0.8518	1.5761	1.9840	5.0000	5.0792
P_{20}	0.0435	0.0291	0.0061	0.1990	0.4317	0.8518	1.5761	1.9836	5.0000	5.0793
Exp. smoothed	0.0435	0.0291	0.0062	0.1995	0.4306	0.8511	1.5750	1.9841	5.0000	5.0822
G–J	0.0439	0.0292	0.0062	0.1999	0.4321	0.8528	1.5807	1.9905	4.9985	5.0951
$P(1\dots4)$	0.0439	0.0292	0.0074	0.2002	0.4318	0.8523	1.5807	1.9904	4.9986	5.0953
$P(1\dots5)$	0.0439	0.0293	0.0009	0.1999	0.4328	0.8519	1.5817	1.9920	4.9973	5.0882
$P(1\dots6)$	0.0440	0.0292	0.0166	0.2030	0.4334	0.8513	1.5791	1.9893	5.0122	5.0954
$P(1\dots7)$	0.0439	0.0293	-0.0047	0.1803	0.4253	0.8659	1.5640	1.9944	4.9599	5.0759
$P(1\dots8)$	0.0457	0.0294	0.0082	0.2618	0.4903	0.7879	1.6048	1.9728	5.0850	5.0928
$P(124)$	0.0444	0.0291	0.0065	0.1999	0.4324	0.8521	1.5770	1.9846	4.9973	5.1027
$P(1248)$	0.0439	0.0292	0.0059	0.2000	0.4325	0.8522	1.5801	1.9909	4.9991	5.0915

Table 1-c: Geske-Johnson problems, III.

	Problem No.									
	21	22	23	24	25	26	27	28	29	30
S	40	40	40	40	40	40	40	40	40	40
r	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490
X	45	35	35	35	40	40	40	45	45	45
σ	0.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
T	0.5833	0.0833	0.3333	0.5833	0.0833	0.3333	0.5833	0.0833	0.3333	0.5833
P_1	5.0000	0.0770	0.6866	1.1888	1.2988	2.4275	3.0634	5.0000	5.5288	5.9723
P_2	5.0831	0.0770	0.6881	1.1962	1.3020	2.4493	3.1103	5.0279	5.6310	6.1268
P_3	5.1560	0.0771	0.6903	1.2023	1.3041	2.4585	3.1272	5.0410	5.6581	6.1683
P_4	5.1892	0.0771	0.6917	1.2059	1.3054	2.4639	3.1366	5.0468	5.6703	6.1872
P_5	5.2077	0.0771	0.6926	1.2083	1.3062	2.4673	3.1427	5.0499	5.6773	6.1982
P_6	5.2192	0.0772	0.6933	1.2100	1.3068	2.4696	3.1469	5.0518	5.6818	6.2055
P_7	5.2271	0.0772	0.6938	1.2111	1.3072	2.4714	3.1500	5.0530	5.6852	6.2108
P_8	5.2327	0.0772	0.6941	1.2122	1.3074	2.4728	3.1523	5.0540	5.6876	6.2148
P_9	5.2354	0.0777	0.6945	1.2127	1.3073	2.4739	3.1538	5.0548	5.6888	6.2176
P_{10}	5.2401	0.0772	0.6944	1.2132	1.3075	2.4737	3.1557	5.0563	5.6911	6.2209
P_{11}	5.2434	0.0774	0.6955	1.2137	1.3082	2.4747	3.1554	5.0552	5.6933	6.2230
P_{12}	5.2460	0.0770	0.6959	1.2146	1.3073	2.4762	3.1579	5.0557	5.6929	6.2233
P_{13}	5.2460	0.0770	0.6957	1.2141	1.3088	2.4771	3.1579	5.0556	5.6936	6.2242
P_{14}	5.2481	0.0769	0.6954	1.2149	1.3090	2.4765	3.1603	5.0575	5.6941	6.2275
P_{15}	5.2494	0.0773	0.6955	1.2151	1.3094	2.4772	3.1597	5.0564	5.6958	6.2284
P_{16}	5.2503	0.0776	0.6951	1.2153	1.3086	2.4777	3.1606	5.0577	5.6971	6.2288
P_{17}	5.2508	0.0777	0.6957	1.2156	1.3078	2.4770	3.1607	5.0569	5.6968	6.2307
P_{18}	5.2529	0.0772	0.6958	1.2147	1.3093	2.4786	3.1622	5.0560	5.6974	6.2304
P_{19}	5.2529	0.0777	0.6952	1.2165	1.3099	2.4780	3.1614	5.0571	5.6988	6.2306
P_{20}	5.2541	0.0771	0.6945	1.2172	1.3099	2.4783	3.1627	5.0561	5.7000	6.2324
Exp. smoothed	5.2496	0.0773	0.6945	1.2157	1.3090	2.4780	3.1622	5.0576	5.7001	6.2314
G–J	5.2719	0.0774	0.6969	1.2194	1.3100	2.4817	3.1733	5.0599	5.7012	6.2365
$P(1\ldots4)$	5.2713	0.0775	0.6967	1.2187	1.3102	2.4834	3.1715	5.0599	5.7011	6.2357
$P(1\ldots5)$	5.2667	0.0769	0.6987	1.2202	1.3095	2.4833	3.1696	5.0589	5.7045	6.2455
$P(1\ldots6)$	5.2611	0.0787	0.6932	1.2190	1.3146	2.4780	3.1700	5.0594	5.7039	6.2372
$P(1\ldots7)$	5.2800	0.0740	0.7103	1.2101	1.2814	2.4998	3.1655	5.0516	5.7282	6.2673
$P(1\ldots8)$	5.2348	0.0963	0.6300	1.2822	1.3544	2.4462	3.1974	5.1220	5.6217	6.2066
$P(124)$	5.2848	0.0772	0.6970	1.2196	1.3100	2.4809	3.1649	5.0622	5.7016	6.2365
$P(1248)$	5.2679	0.0775	0.6971	1.2194	1.3096	2.4830	3.1704	5.0595	5.7037	6.2412

Table 1-d: Geske-Johnson problems, IV.

	Problem No.								
	31	32	33	34	35	36	37	38	39
S	40	40	40	40	40	40	40	40	40
r	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490	0.0490
X	35	35	35	40	40	40	45	45	45
σ	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
T	0.0833	0.3333	0.5833	0.0833	0.3333	0.5833	0.0833	0.3333	0.5833
P_1	0.2456	1.3297	2.1127	1.7575	3.3336	4.2473	5.2360	6.3767	7.1654
P_2	0.2457	1.3324	2.1234	1.7601	3.3531	4.2906	5.2655	6.4473	7.2810
P_3	0.2458	1.3357	2.1317	1.7623	3.3625	4.3081	5.2731	6.4672	7.3135
P_4	0.2458	1.3378	2.1366	1.7635	3.3681	4.3181	5.2765	6.4772	7.3297
P_5	0.2460	1.3391	2.1397	1.7644	3.3717	4.3245	5.2785	6.4833	7.3397
P_6	0.2461	1.3400	2.1419	1.7649	3.3741	4.3289	5.2799	6.4876	7.3466
P_7	0.2461	1.3407	2.1436	1.7654	3.3760	4.3320	5.2808	6.4907	7.3517
P_8	0.2461	1.3414	2.1448	1.7657	3.3773	4.3346	5.2816	6.4930	7.3555
P_9	0.2458	1.3423	2.1464	1.7657	3.3788	4.3366	5.2813	6.4934	7.3582
P_{10}	0.2466	1.3426	2.1473	1.7659	3.3791	4.3392	5.2826	6.4957	7.3613
P_{11}	0.2467	1.3428	2.1479	1.7660	3.3793	4.3393	5.2823	6.4985	7.3631
P_{12}	0.2462	1.3426	2.1483	1.7662	3.3795	4.3403	5.2847	6.4990	7.3658
P_{13}	0.2468	1.3428	2.1488	1.7660	3.3814	4.3409	5.2834	6.5004	7.3655
P_{14}	0.2462	1.3422	2.1485	1.7668	3.3815	4.3420	5.2827	6.5001	7.3682
P_{15}	0.2465	1.3443	2.1497	1.7666	3.3825	4.3426	5.2826	6.5003	7.3676
P_{16}	0.2461	1.3423	2.1496	1.7657	3.3813	4.3433	5.2843	6.5013	7.3680
P_{17}	0.2464	1.3442	2.1504	1.7665	3.3825	4.3434	5.2839	6.5034	7.3703
P_{18}	0.2458	1.3439	2.1513	1.7675	3.3827	4.3438	5.2847	6.5017	7.3698
P_{19}	0.2475	1.3442	2.1515	1.7678	3.3822	4.3444	5.2840	6.5034	7.3717
P_{20}	0.2468	1.3438	2.1500	1.7679	3.3828	4.3454	5.2852	6.5022	7.3713
Exp. smoothed	0.2465	1.3441	2.1508	1.7670	3.3821	4.3427	5.2836	6.5003	7.3705
G–J	0.2466	1.3450	2.1568	1.7679	3.3632	4.3556	5.2855	6.5093	7.3831
$P(1\ldots4)$	0.2461	1.3454	2.1531	1.7680	3.3885	4.3555	5.2860	6.5095	7.3812
$P(1\ldots5)$	0.2483	1.3455	2.1548	1.7683	3.3880	4.3518	5.2856	6.5108	7.3853
$P(1\ldots6)$	0.2447	1.3415	2.1553	1.7665	3.3864	4.3549	5.2934	6.5143	7.3846
$P(1\ldots7)$	0.2395	1.3616	2.1514	1.7760	3.3959	4.3273	5.2700	6.4985	7.3931
$P(1\ldots8)$	0.2735	1.3497	2.1598	1.7548	3.3099	4.4861	5.3345	6.5257	7.3464
$P(124)$	0.2461	1.3458	2.1549	1.7683	3.3866	4.3494	5.2849	6.5034	7.3724
$P(1248)$	0.2466	1.3456	2.1541	1.7682	3.3877	4.3535	5.2867	6.5104	7.3838

Table 2-a: Results for the problems of Table 2 in Broadie and Detemple's (1996) paper, I.

	Problem No.									
	1	2	3	4	5	6	7	8	9	10
S	80	90	100	110	120	80	90	100	110	120
r	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.4	0.4	0.4	0.4	0.4
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
C_1	2.2407	4.3546	7.3859	11.3310	16.1172	10.3094	14.1616	18.5322	23.3629	28.5983
C_2	2.3776	4.7592	8.3012	13.0361	18.8690	10.7921	14.9743	19.7805	25.1508	31.0234
C_3	2.4319	4.8823	8.5709	13.5753	19.8146	10.9476	15.2071	20.1208	25.6356	31.6942
C_4	2.4638	4.9458	8.6950	13.8240	20.2768	11.0331	15.3256	20.2817	25.8538	31.9895
C_5	2.4847	4.9859	8.7668	13.9631	20.5450	11.0881	15.4000	20.3787	25.9794	32.1536
C_6	2.4994	5.0139	8.8144	14.0506	20.7175	11.1260	15.4512	20.4448	26.0625	32.2587
C_7	2.5100	5.0345	8.8483	14.1103	20.8369	11.1534	15.4889	20.4931	26.1221	32.3325
C_8	2.5182	5.0503	8.8743	14.1536	20.9235	11.1744	15.5172	20.5297	26.1674	32.3875
C_9	2.5243	5.0623	8.8945	14.1863	20.9880	11.1911	15.5398	20.5584	26.2014	32.4321
C_{10}	2.5294	5.0730	8.9108	14.2125	21.0416	11.2042	15.5587	20.5819	26.2322	32.4649
C_{11}	2.5333	5.0816	8.9238	14.2351	21.0800	11.2152	15.5726	20.6004	26.2558	32.4949
C_{12}	2.5377	5.0896	8.9354	14.2512	21.1138	11.2232	15.5847	20.6183	26.2754	32.5179
C_{13}	2.5423	5.0946	8.9470	14.2669	21.1393	11.2322	15.5948	20.6315	26.2901	32.5348
C_{14}	2.5438	5.0984	8.9543	14.2780	21.1630	11.2394	15.6034	20.6427	26.3084	32.5555
C_{15}	2.5451	5.1022	8.9613	14.2882	21.1814	11.2442	15.6124	20.6497	26.3186	32.5693
C_{16}	2.5490	5.1084	8.9673	14.2957	21.1934	11.2476	15.6195	20.6596	26.3300	32.5818
C_{17}	2.5480	5.1100	8.9726	14.3045	21.2088	11.2533	15.6243	20.6684	26.3407	32.5924
C_{18}	2.5527	5.1122	8.9791	14.3145	21.2209	11.2572	15.6321	20.6761	26.3508	32.6060
C_{19}	2.5510	5.1181	8.9828	14.3203	21.2343	11.2589	15.6349	20.6810	26.3571	32.6151
C_{20}	2.5551	5.1201	8.9881	14.3262	21.2453	11.2631	15.6409	20.6865	26.3642	32.6263
Exp. smoothed	2.5531	5.1214	8.9908	14.3246	21.2415	11.2649	15.6423	20.6896	26.3683	32.4997
'True' values	2.5800	5.1670	9.0660	14.4430	21.4140	11.3260	15.7220	20.7930	26.4950	32.7810
$C(1\ldots4)$	2.5875	5.1551	9.0134	14.4454	21.5784	11.3437	15.7135	20.7406	26.4021	32.6765
$C(1\ldots5)$	2.5821	5.1788	9.0483	14.4072	21.4820	11.3458	15.7534	20.8132	26.4779	32.7203
$C(1\ldots6)$	2.5790	5.1736	9.0771	14.4055	21.4349	11.3143	15.7116	20.8199	26.5234	32.7799
$C(1\ldots7)$	2.5533	5.1559	9.0371	14.4130	21.4649	11.2800	15.7957	20.8274	26.4947	32.8260
$C(1\ldots8)$	2.6649	5.2134	9.2526	14.4626	21.2436	11.5417	15.4307	20.6523	26.6471	32.7397
$C(124)$	2.5618	5.1220	9.0462	14.5688	21.7059	11.2739	15.6401	20.7010	26.4295	32.7913
$C(1248)$	2.5827	5.1680	9.0412	14.4219	21.5073	11.3375	15.7309	20.7867	26.4596	32.7199

Table 2-b: Results for the problems of Table 2 in Broadie and Detemple's (1996) paper, II.

	Problem No.									
	11	12	13	14	15	16	17	18	19	20
S	80	90	100	110	120	80	90	100	110	120
r	0.0000	0.0000	0.0000	0.0000	0.0000	0.0700	0.0700	0.0700	0.0700	0.0700
X	100	100	100	100	100	100	100	100	100	100
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0300	0.0300	0.0300	0.0300	0.0300
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
C_1	4.6444	7.2687	10.5418	14.4296	18.8821	12.1329	17.3427	23.3007	29.8818	36.9725
C_2	5.0520	8.0791	11.9437	16.6148	22.0275	12.1351	17.3491	23.3160	29.9137	37.0319
C_3	5.1848	8.3221	12.3664	17.3039	23.0743	12.1386	17.3560	23.3279	29.9317	37.0573
C_4	5.2570	8.4410	12.5619	17.6219	23.5719	12.1403	17.3596	23.3335	29.9406	37.0702
C_5	5.3036	8.5138	12.6746	17.8006	23.8545	12.1414	17.3613	23.3369	29.9456	37.0776
C_6	5.3365	8.5642	12.7488	17.9141	24.0336	12.1421	17.3626	23.3388	29.9490	37.0824
C_7	5.3607	8.6011	12.8020	17.9922	24.1558	12.1426	17.3635	23.3405	29.9510	37.0855
C_8	5.3788	8.6297	12.8421	18.0496	24.2438	12.1430	17.3641	23.3415	29.9528	37.0882
C_9	5.3937	8.6520	12.8747	18.0958	24.3103	12.1433	17.3642	23.3427	29.9540	37.0897
C_{10}	5.4064	8.6699	12.8991	18.1291	24.3627	12.1434	17.3647	23.3432	29.9559	37.0914
C_{11}	5.4161	8.6862	12.9201	18.1563	24.4037	12.1437	17.3659	23.3445	29.9569	37.0931
C_{12}	5.4225	8.6974	12.9388	18.1819	24.4367	12.1440	17.3652	23.3434	29.9568	37.0935
C_{13}	5.4307	8.7086	12.9544	18.2018	24.4667	12.1441	17.3660	23.3443	29.9571	37.0928
C_{14}	5.4368	8.7184	12.9669	18.2185	24.4884	12.1437	17.3658	23.3443	29.9573	37.0958
C_{15}	5.4411	8.7266	12.9789	18.2327	24.5078	12.1448	17.3666	23.3459	29.9587	37.0958
C_{16}	5.4460	8.7331	12.9887	18.2467	24.5253	12.1443	17.3656	23.3462	29.9598	37.0970
C_{17}	5.4509	8.7410	12.9965	18.2582	24.5387	12.1438	17.3666	23.3450	29.9601	37.0958
C_{18}	5.4525	8.7422	13.0049	18.2700	24.5553	12.1446	17.3665	23.3448	29.9563	37.0949
C_{19}	5.4560	8.7499	13.0103	18.2760	24.5681	12.1437	17.3673	23.3449	29.9590	37.0977
C_{20}	5.4603	8.7555	13.0179	18.2833	24.5805	12.1450	17.3655	23.3467	29.9610	37.0974
Exp. smoothed	5.4623	8.7594	13.0235	18.2863	24.5751	12.1443	17.3661	23.3450	29.9583	37.0955
'True' values	5.5180	8.8420	13.1420	18.4530	24.7910	12.1450	17.3690	23.3480	29.9640	37.1040
$C(1\ldots4)$	5.5138	8.7942	13.0652	18.4180	24.8934	12.1444	17.3695	23.3442	29.9618	37.1070
$C(1\ldots5)$	5.5303	8.8444	13.1009	18.3905	24.7790	12.1460	17.3575	23.3554	29.9636	37.1022
$C(1\ldots6)$	5.5320	8.8872	13.1524	18.4041	24.7461	12.1425	17.3913	23.3236	29.9675	37.0986
$C(1\ldots7)$	5.4790	8.7499	13.1774	18.4209	24.7338	12.1602	17.3406	23.4384	29.9152	37.0970
$C(1\ldots8)$	5.3916	9.1701	13.0275	18.5811	24.7014	12.0994	17.3378	23.0717	30.1828	37.2436
$C(124)$	5.4629	8.7740	13.1250	18.5719	25.0974	12.1484	17.3752	23.3575	29.9747	37.1141
$C(1248)$	5.5208	8.8306	13.0999	18.4062	24.8132	12.1452	17.3669	23.3478	29.9627	37.1041

Table 3: Results for the problems of Table 2 in Arciniega and Allen's paper

	Problem No.		
	1	2	3
S	90	100	110
r	0.0300	0.0300	0.0300
X	100	100	100
σ	0.4	0.4	0.4
d	0.0700	0.0700	0.0700
T	0.5000	0.5000	0.5000
C_1	5.6221	10.0211	15.7676
C_2	5.6491	10.1095	15.9815
C_3	5.6686	10.1461	16.0434
C_4	5.6802	10.1671	16.0755
C_5	5.6875	10.1805	16.0954
C_6	5.6929	10.1896	16.1095
C_7	5.6967	10.1964	16.1194
C_8	5.6996	10.2013	16.1271
C_9	5.7020	10.2055	16.1329
C_{10}	5.7038	10.2087	16.1377
C_{11}	5.7055	10.2112	16.1415
C_{12}	5.7065	10.2133	16.1448
C_{13}	5.7080	10.2153	16.1477
C_{14}	5.7088	10.2170	16.1500
C_{15}	5.7095	10.2183	16.1522
C_{16}	5.7104	10.2197	16.1539
C_{17}	5.7111	10.2208	16.1552
C_{18}	5.7114	10.2216	16.1568
C_{19}	5.7120	10.2226	16.1581
C_{20}	5.7126	10.2234	16.1596
Exp. smoothed	5.7110	10.2205	16.1597
'True' value	5.7220	10.2390	16.1810
$C(1\ldots4)$	5.7223	10.2447	16.1831
$C(1\ldots5)$	5.7157	10.2385	16.1843
$C(1\ldots6)$	5.7425	10.2275	16.2085
$C(1\ldots7)$	5.6674	10.2893	16.0858
$C(1\ldots8)$	5.8389	10.0333	16.4124
$C(124)$	5.7231	10.2337	16.1608
$C(1248)$	5.7213	10.2400	16.1847

Table 4-a: Results for the problems of Table 1 in Ju's paper (the same as those of Table 1 in Broadie–Detemple's (1996) paper), I.

	Problem No.									
	1	2	3	4	5	6	7	8	9	10
S	80	90	100	110	120	80	90	100	110	120
r	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300	0.0300
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.4	0.4	0.4	0.4	0.4
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700
T	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C_1	0.2148	1.3451	4.5778	10.4207	20.0000	2.6506	5.6221	10.0211	15.7676	22.6502
C_2	0.2150	1.3538	4.6734	10.8164	20.0000	2.6561	5.6491	10.1095	15.9815	23.0637
C_3	0.2156	1.3620	4.7051	10.9267	20.0000	2.6639	5.6687	10.1461	16.0434	23.1707
C_4	0.2164	1.3670	4.7225	10.9754	20.0000	2.6687	5.6803	10.1671	16.0755	23.2187
C_5	0.2168	1.3702	4.7335	11.0021	20.0000	2.6719	5.6876	10.1805	16.0956	23.2465
C_6	0.2170	1.3726	4.7413	11.0189	20.0000	2.6744	5.6928	10.1897	16.1094	23.2649
C_7	0.2173	1.3743	4.7470	11.0307	20.0000	2.6760	5.6966	10.1963	16.1193	23.2781
C_8	0.2174	1.3756	4.7513	11.0390	20.0000	2.6773	5.6995	10.2013	16.1271	23.2882
C_9	0.2179	1.3767	4.7547	11.0459	20.0000	2.6784	5.7014	10.2049	16.1338	23.2957
C_{10}	0.2180	1.3783	4.7582	11.0507	20.0000	2.6792	5.7040	10.2090	16.1380	23.3029
C_{11}	0.2175	1.3784	4.7600	11.0544	20.0000	2.6800	5.7052	10.2121	16.1436	23.3045
C_{12}	0.2180	1.3796	4.7624	11.0604	20.0000	2.6809	5.7077	10.2144	16.1428	23.3138
C_{13}	0.2182	1.3790	4.7641	11.0617	20.0000	2.6807	5.7069	10.2151	16.1482	23.3132
C_{14}	0.2174	1.3792	4.7639	11.0644	20.0000	2.6828	5.7092	10.2160	16.1519	23.3168
C_{15}	0.2180	1.3812	4.7687	11.0677	20.0000	2.6829	5.7099	10.2190	16.1520	23.3213
C_{16}	0.2182	1.3813	4.7655	11.0661	20.0000	2.6827	5.7108	10.2217	16.1549	23.3256
C_{17}	0.2191	1.3810	4.7663	11.0712	20.0000	2.6828	5.7108	10.2207	16.1563	23.3275
C_{18}	0.2188	1.3804	4.7689	11.0703	20.0000	2.6838	5.7113	10.2218	16.1550	23.3303
C_{19}	0.2189	1.3820	4.7690	11.0749	20.0000	2.6840	5.7128	10.2237	16.1580	23.3309
C_{20}	0.2173	1.3799	4.7699	11.0739	20.0000	2.6828	5.7127	10.2237	16.1613	23.3303
Exp. smoothed	0.2185	1.3809	4.7702	11.0740	20.0000	2.6835	5.7104	10.2211	16.1601	23.3328
'True' values	0.2194	1.3864	4.7825	11.0978	20.0004	2.6889	5.7223	10.2385	16.1812	23.3598
Ju's results	0.2196	1.3872	4.7837	11.0993	20.0005	2.6899	5.7237	10.2404	16.1831	23.3622
$C(1\dots4)$	0.2212	1.3852	4.7853	11.0891	20.0078	2.6863	5.7217	10.2446	16.1835	23.3411
$C(1\dots5)$	0.2158	1.3838	4.7799	11.0848	20.0041	2.6888	5.7158	10.2399	16.1876	23.3616
$C(1\dots6)$	0.2191	1.3955	4.8036	11.1016	20.0032	2.6929	5.7357	10.2290	16.1838	23.3585
$C(1\dots7)$	0.2279	1.3704	4.7349	11.1078	20.0029	2.6491	5.6901	10.2430	16.1579	23.3749
$C(1\dots8)$	0.1810	1.3460	4.8351	10.9405	20.0771	2.8248	5.7940	10.2627	16.3131	23.3733
$C(124)$	0.2186	1.3860	4.7725	11.1086	20.0125	2.6879	5.7233	10.2336	16.1608	23.3391
$C(1248)$	0.2188	1.3854	4.7842	11.0895	20.0058	2.6875	5.7208	10.2401	16.1848	23.3542

Table 4-b: Results for the problems of Table 1 in Ju's paper (the same as those of Table 1 in Broadie–Detemple's (1996) paper), II.

	Problem No.									
	11	12	13	14	15	16	17	18	19	20
S	80	90	100	110	120	80	90	100	110	120
r	0.0000	0.0000	0.0000	0.0000	0.0000	0.0700	0.0700	0.0700	0.0700	0.0700
X	100	100	100	100	100	100	100	100	100	100
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0700	0.0700	0.0700	0.0700	0.0700	0.0300	0.0300	0.0300	0.0300	0.0300
T	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C_1	1.0064	3.0041	6.6943	12.1661	20.0000	1.6644	4.4947	9.2506	15.7975	23.7062
C_2	1.0106	3.0405	6.8562	12.6149	20.0449	1.6644	4.4947	9.2506	15.7975	23.7062
C_3	1.0167	3.0622	6.9086	12.7383	20.3122	1.6644	4.4947	9.2506	15.7975	23.7055
C_4	1.0205	3.0753	6.9369	12.7941	20.4347	1.6644	4.4947	9.2509	15.7999	23.7061
C_5	1.0232	3.0837	6.9550	12.8259	20.5030	1.6644	4.4947	9.2512	15.7975	23.7061
C_6	1.0250	3.0896	6.9676	12.8469	20.5458	1.6644	4.4947	9.2514	15.7935	23.7062
C_7	1.0264	3.0938	6.9769	12.8616	20.5749	1.6644	4.4948	9.2501	15.7975	23.7062
C_8	1.0277	3.0972	6.9840	12.8729	20.5954	1.6644	4.4948	9.2506	15.7973	23.7062
C_9	1.0281	3.0997	6.9904	12.8830	20.6110	1.6644	4.4949	9.2502	15.7976	23.7067
C_{10}	1.0291	3.1032	6.9938	12.8892	20.6221	1.6644	4.4948	9.2508	15.7970	23.7057
C_{11}	1.0298	3.1038	6.9980	12.8969	20.6320	1.6644	4.4947	9.2491	15.7974	23.7067
C_{12}	1.0305	3.1072	7.0006	12.8989	20.6400	1.6644	4.4945	9.2503	15.7974	23.7070
C_{13}	1.0308	3.1072	7.0031	12.9026	20.6454	1.6644	4.4948	9.2508	15.7977	23.7069
C_{14}	1.0311	3.1072	7.0053	12.9068	20.6500	1.6644	4.4950	9.2504	15.7972	23.7060
C_{15}	1.0320	3.1080	7.0096	12.9108	20.6587	1.6644	4.4949	9.2503	15.7972	23.7060
C_{16}	1.0322	3.1090	7.0078	12.9126	20.6600	1.6644	4.4948	9.2492	15.7975	23.7065
C_{17}	1.0344	3.1085	7.0111	12.9152	20.6631	1.6644	4.4949	9.2505	15.7966	23.7067
C_{18}	1.0321	3.1119	7.0113	12.9149	20.6674	1.6644	4.4947	9.2507	15.7978	23.7062
C_{19}	1.0319	3.1107	7.0120	12.9207	20.6682	1.6644	4.4949	9.2538	15.7974	23.7059
C_{20}	1.0332	3.1140	7.0169	12.9240	20.6700	1.6644	4.4947	9.2513	15.7974	23.7067
Exp. smoothed	1.0329	3.1104	7.0106	12.9219	20.6657	1.6644	4.4948	9.2507	15.7973	23.7063
'True' values	1.0373	3.1233	7.0354	12.9552	20.7173	1.6644	4.4947	9.2504	15.7977	23.7061
Ju's results	1.0381	3.1247	7.0371	12.9574	20.7194	1.6644	4.4947	9.2506	15.7975	23.7062
$C(1\ldots4)$	1.0337	3.1239	7.0374	12.9346	20.7425	1.6644	4.4947	9.2536	15.8228	23.7136
$C(1\ldots5)$	1.0415	3.1208	7.0385	12.9457	20.7193	1.6644	4.4955	9.2532	15.6970	23.6965
$C(1\ldots6)$	1.0247	3.1231	7.0426	12.9660	20.7078	1.6642	4.4945	9.2521	15.7387	23.7199
$C(1\ldots7)$	1.0535	3.1078	7.0169	12.9105	20.7385	1.6658	4.4942	9.0140	17.0787	23.6793
$C(1\ldots8)$	1.1109	3.2148	7.1118	13.1302	20.5338	1.6584	4.4989	10.4784	11.3676	23.7361
$C(124)$	1.0355	3.1210	7.0174	12.9431	20.7878	1.6644	4.4947	9.2514	15.8038	23.7059
$C(1248)$	1.0367	3.1224	7.0382	12.9447	20.7256	1.6644	4.4950	9.2499	15.7906	23.7064

Table 5-a: Results for the problems of Table 2 in Ju's paper (the same as those of Table 5 in Barone–Adesi and Whaley's paper), I.

	Problem No.									
	1	2	3	4	5	6	7	8	9	10
S	80	90	100	110	120	80	90	100	110	120
r	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
d	0.1200	0.1200	0.1200	0.1200	0.1200	0.0800	0.0800	0.0800	0.0800	0.0800
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
P_1	24.7773	19.6199	15.2522	11.6711	8.8138	19.5248	14.6756	10.8169	7.8472	5.6218
P_2	25.3631	19.8972	15.3757	11.7237	8.8355	21.1375	15.6031	11.3139	8.0990	5.7441
P_3	25.4777	19.9621	15.4190	11.7527	8.8540	21.5868	15.8354	11.4432	8.1805	5.7991
P_4	25.5263	19.9949	15.4413	11.7670	8.8628	21.7813	15.9354	11.5064	8.2254	5.8306
P_5	25.5543	20.0145	15.4542	11.7750	8.8679	21.8854	15.9911	11.5453	8.2535	5.8500
P_6	25.5728	20.0273	15.4623	11.7802	8.8709	21.9489	16.0275	11.5719	8.2726	5.8631
P_7	25.5861	20.0363	15.4682	11.7839	8.8734	21.9911	16.0533	11.5910	8.2864	5.8726
P_8	25.5959	20.0429	15.4727	11.7867	8.8751	22.0212	16.0726	11.6056	8.2965	5.8799
P_9	25.6040	20.0489	15.4758	11.7885	8.8764	22.0436	16.0872	11.6153	8.3044	5.8844
P_{10}	25.6103	20.0517	15.4770	11.7907	8.8774	22.0608	16.1003	11.6262	8.3109	5.8889
P_{11}	25.6128	20.0532	15.4817	11.7909	8.8787	22.0734	16.1107	11.6345	8.3179	5.8943
P_{12}	25.6186	20.0561	15.4816	11.7916	8.8771	22.0875	16.1164	11.6396	8.3197	5.8983
P_{13}	25.6215	20.0613	15.4801	11.7943	8.8807	22.0947	16.1249	11.6445	8.3236	5.8981
P_{14}	25.6254	20.0621	15.4843	11.7955	8.8813	22.1008	16.1310	11.6480	8.3259	5.9020
P_{15}	25.6278	20.0629	15.4855	11.7941	8.8800	22.1093	16.1388	11.6521	8.3291	5.9022
P_{16}	25.6336	20.0644	15.4846	11.7960	8.8815	22.1148	16.1419	11.6576	8.3343	5.9063
P_{17}	25.6304	20.0651	15.4879	11.7967	8.8807	22.1230	16.1460	11.6591	8.3350	5.9080
P_{18}	25.6303	20.0681	15.4875	11.7950	8.8810	22.1260	16.1458	11.6620	8.3341	5.9073
P_{19}	25.6343	20.0673	15.4904	11.7959	8.8814	22.1312	16.1522	11.6633	8.3393	5.9072
P_{20}	25.6368	20.0686	15.4863	11.7970	8.8824	22.1326	16.1540	11.6661	8.3431	5.9093
Exp. smoothed	25.6346	20.0675	15.4857	11.7948	8.8805	22.1287	16.1546	11.6663	8.3411	5.9118
'True' values	25.6577	20.0832	15.4981	11.8032	8.8856	22.2050	16.2071	11.7037	8.3671	5.9299
Ju's results	25.6570	20.0817	15.4970	11.8022	8.8850	22.2084	16.2106	11.7066	8.3695	5.9323
$P(1\ldots4)$	25.6537	20.1084	15.5116	11.8032	8.8810	22.2083	16.1654	11.7047	8.3892	5.9437
$P(1\ldots5)$	25.6767	20.0888	15.4934	11.7967	8.8866	22.1757	16.1969	11.7257	8.3708	5.9232
$P(1\ldots6)$	25.6685	20.0706	15.4870	11.8118	8.8723	22.1861	16.2360	11.7101	8.3618	5.9293
$P(1\ldots7)$	25.6564	20.1139	15.5541	11.8160	8.9634	22.1821	16.1966	11.6621	8.3920	5.9394
$P(1\ldots8)$	25.6340	20.0484	15.5113	11.7865	8.6922	22.2679	16.2153	11.9086	8.2090	5.9764
$P(124)$	25.6030	20.0653	15.5094	11.8218	8.9010	22.3167	16.1800	11.6616	8.3519	5.9338
$P(1248)$	25.6653	20.0940	15.5023	11.8027	8.8845	22.1907	16.1921	11.7132	8.3760	5.9332

Table 5-b: Results for the problems of Table 2 in Ju's paper (the same as those of Table 5 in Barone–Adesi and Whaley's paper), II.

	Problem No.									
	11	12	13	14	15	16	17	18	19	20
S	80	90	100	110	120	80	90	100	110	120
r	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
X	100	100	100	100	100	100	100	100	100	100
σ	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0000	0.0000	0.0000	0.0000	0.0000
T	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
P_1	20.0000	10.3263	7.1676	4.9015	3.3153	20.0000	10.0000	4.4061	2.8258	1.7969
P_2	20.0000	12.0493	8.1382	5.4163	3.5759	20.0000	10.0000	5.6692	3.4858	2.1262
P_3	20.0000	12.6056	8.4254	5.5699	3.6654	20.0000	10.0000	6.1178	3.7027	2.2353
P_4	20.0000	12.8647	8.5571	5.6470	3.7159	20.0000	10.4237	6.3350	3.8080	2.2934
P_5	20.0000	13.0100	8.6326	5.6954	3.7489	20.0000	10.7064	6.4603	3.8709	2.3309
P_6	20.0000	13.1014	8.6824	5.7289	3.7718	20.0000	10.8932	6.5405	3.9138	2.3574
P_7	20.0000	13.1635	8.7178	5.7534	3.7887	20.0000	11.0246	6.5964	3.9448	2.3772
P_8	20.0000	13.2083	8.7447	5.7725	3.8015	20.0000	11.1215	6.6377	3.9690	2.3927
P_9	20.0000	13.2416	8.7645	5.7865	3.8099	20.0000	11.1947	6.6681	3.9883	2.4041
P_{10}	20.0000	13.2680	8.7844	5.7998	3.8189	20.0000	11.2542	6.6942	4.0043	2.4165
P_{11}	20.0000	13.2899	8.7964	5.8087	3.8272	20.0000	11.3017	6.7140	4.0190	2.4232
P_{12}	20.0000	13.3054	8.8091	5.8181	3.8369	20.0000	11.3401	6.7325	4.0282	2.4308
P_{13}	20.0000	13.3221	8.8184	5.8240	3.8371	20.0000	11.3716	6.7451	4.0364	2.4381
P_{14}	20.0281	13.3323	8.8262	5.8306	3.8425	20.0000	11.3945	6.7585	4.0448	2.4428
P_{15}	20.0534	13.3457	8.8365	5.8362	3.8442	20.0000	11.4199	6.7712	4.0505	2.4423
P_{16}	20.0746	13.3521	8.8416	5.8414	3.8483	20.0000	11.4402	6.7806	4.0591	2.4533
P_{17}	20.0974	13.3625	8.8469	5.8465	3.8513	20.0000	11.4524	6.7869	4.0627	2.4542
P_{18}	20.1135	13.3709	8.8539	5.8501	3.8526	20.0000	11.4701	6.7974	4.0677	2.4535
P_{19}	20.1264	13.3750	8.8558	5.8501	3.8572	20.0000	11.4811	6.8044	4.0741	2.4584
P_{20}	20.1414	13.3831	8.8637	5.8555	3.8543	20.0000	11.4968	6.8099	4.0756	2.4605
Exp. smoothed	20.0338	13.3803	8.8655	5.8582	3.8591	20.0000	11.5934	6.8157	4.0821	2.4644
'True' values	20.3500	13.4968	8.9438	5.9119	3.8975	20.0000	11.6974	6.9320	4.1550	2.5102
Ju's results	20.3511	13.5000	8.9474	5.9146	3.8997	20.0000	11.6991	6.9346	4.1571	2.5119
$P(1\ldots4)$	20.5123	13.5246	8.8909	5.8901	3.9042	19.7315	11.8840	6.9263	4.1039	2.4916
$P(1\ldots5)$	20.4568	13.4702	8.9181	5.9274	3.9086	19.8255	11.7968	6.8916	4.1333	2.5184
$P(1\ldots6)$	20.4286	13.4586	8.9556	5.9188	3.8828	19.8851	11.7335	6.8717	4.1831	2.5145
$P(1\ldots7)$	20.3666	13.4804	8.9220	5.8936	3.9320	19.8651	11.6892	6.9800	4.0767	2.5156
$P(1\ldots8)$	20.5382	13.4704	9.0945	6.0700	3.7601	20.0703	11.7581	6.8590	4.5005	2.5993
$P(124)$	20.5755	13.6494	8.9315	5.8599	3.8624	19.5135	11.9430	7.0237	4.1248	2.4624
$P(1248)$	20.4686	13.4891	8.9160	5.9112	3.9025	19.8045	11.8126	6.9054	4.1308	2.5081

Table 6-a: Results for new problems (Call options, $r > d$), I.

	Problem No.							
	1	2	3	4	5	6	7	8
S	70	70	70	70	70	70	70	70
r	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600
X	60	60	60	60	60	70	70	70
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	0.2500	0.5000	1.0000	2.0000	3.0000	0.2500	0.5000	1.0000
C_1	10.9067	12.0691	13.9497	16.5988	18.4211	4.3081	6.1147	8.6232
C_2	10.9068	12.0715	13.9672	16.6870	18.6291	4.3081	6.1149	8.6265
C_3	10.9069	12.0726	13.9734	16.7124	18.6825	4.3081	6.1151	8.6293
C_4	10.9070	12.0732	13.9765	16.7256	18.7097	4.3081	6.1151	8.6305
C_5	10.9072	12.0735	13.9783	16.7334	18.7261	4.3081	6.1153	8.6312
C_6	10.9070	12.0737	13.9795	16.7385	18.7369	4.3082	6.1153	8.6317
C_7	10.9070	12.0739	13.9804	16.7420	18.7444	4.3081	6.1154	8.6321
C_8	10.9072	12.0739	13.9809	16.7447	18.7501	4.3081	6.1153	8.6323
C_9	10.9059	12.0742	13.9822	16.7478	18.7530	4.3076	6.1158	8.6323
C_{10}	10.9068	12.0739	13.9810	16.7479	18.7568	4.3079	6.1154	8.6331
C_{11}	10.9073	12.0741	13.9833	16.7502	18.7605	4.3086	6.1157	8.6335
C_{12}	10.9077	12.0731	13.9822	16.7513	18.7630	4.3072	6.1152	8.6328
C_{13}	10.9069	12.0752	13.9822	16.7521	18.7646	4.3083	6.1151	8.6326
C_{14}	10.9068	12.0746	13.9830	16.7521	18.7663	4.3082	6.1154	8.6329
C_{15}	10.9073	12.0739	13.9821	16.7518	18.7681	4.3083	6.1157	8.6333
C_{16}	10.9070	12.0739	13.9827	16.7529	18.7699	4.3082	6.1156	8.6323
C_{17}	10.9069	12.0739	13.9820	16.7555	18.7694	4.3081	6.1148	8.6334
C_{18}	10.9073	12.0740	13.9843	16.7539	18.7707	4.3083	6.1152	8.6319
C_{19}	10.9080	12.0749	13.9835	16.7544	18.7719	4.3081	6.1155	8.6334
C_{20}	10.9078	12.0747	13.9836	16.7539	18.7718	4.3082	6.1157	8.6340
Exp. smoothed	10.9072	12.0743	13.9829	16.7533	18.7720	4.3081	6.1154	8.6330
$C(1\dots4)$	10.9067	12.0742	13.9856	16.7707	18.8017	4.3080	6.1145	8.6327
$C(1\dots5)$	10.9116	12.0760	13.9840	16.7576	18.7904	4.3084	6.1197	8.6324
$C(1\dots6)$	10.8832	12.0743	13.9849	16.7669	18.7809	4.3105	6.1052	8.6436
$C(1\dots7)$	10.9759	12.0726	14.0121	16.7453	18.7877	4.2851	6.1401	8.6101
$C(1\dots8)$	10.8495	12.0274	13.8221	16.8145	18.8055	4.3978	6.0329	8.6706
$C(124)$	10.9072	12.0751	13.9862	16.7606	18.7745	4.3082	6.1155	8.6362
$C(1248)$	10.9076	12.0745	13.9850	16.7640	18.7929	4.3081	6.1156	8.6336

Table 6-b: Results for new problems (Call options, $r > d$), II.

	Problem No.						
	9	10	11	12	13	14	15
S	70	70	70	70	70	70	70
r	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600
X	70	70	80	80	80	80	80
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	2.0000	3.0000	0.2500	0.5000	1.0000	2.0000	3.0000
C_1	11.9562	14.2442	1.2021	2.6793	5.0546	8.4913	10.9725
C_2	11.9885	14.3429	1.2021	2.6793	5.0552	8.5027	11.0190
C_3	12.0045	14.3796	1.2021	2.6793	5.0562	8.5124	11.0450
C_4	12.0128	14.3992	1.2018	2.6793	5.0567	8.5173	11.0586
C_5	12.0174	14.4107	1.2021	2.6793	5.0568	8.5199	11.0665
C_6	12.0205	14.4181	1.2020	2.6793	5.0571	8.5217	11.0718
C_7	12.0225	14.4235	1.2020	2.6793	5.0571	8.5230	11.0752
C_8	12.0241	14.4273	1.2021	2.6794	5.0573	8.5241	11.0779
C_9	12.0254	14.4299	1.2019	2.6799	5.0579	8.5249	11.0799
C_{10}	12.0271	14.4325	1.2019	2.6794	5.0572	8.5248	11.0818
C_{11}	12.0271	14.4347	1.2023	2.6792	5.0576	8.5257	11.0831
C_{12}	12.0267	14.4352	1.2018	2.6794	5.0581	8.5260	11.0840
C_{13}	12.0272	14.4364	1.2018	2.6797	5.0577	8.5267	11.0846
C_{14}	12.0285	14.4376	1.2018	2.6798	5.0573	8.5277	11.0862
C_{15}	12.0298	14.4388	1.2019	2.6795	5.0584	8.5262	11.0862
C_{16}	12.0284	14.4396	1.2019	2.6794	5.0584	8.5269	11.0872
C_{17}	12.0299	14.4408	1.2018	2.6787	5.0581	8.5266	11.0874
C_{18}	12.0295	14.4419	1.2019	2.6793	5.0573	8.5280	11.0872
C_{19}	12.0297	14.4409	1.2025	2.6794	5.0572	8.5279	11.0870
C_{20}	12.0296	14.4432	1.2028	2.6797	5.0576	8.5279	11.0876
Exp. smoothed	12.0288	14.4426	1.2020	2.6794	5.0579	8.5272	11.0888
$C(1\ldots4)$	12.0368	14.4645	1.1993	2.6784	5.0576	8.5291	11.0977
$C(1\ldots5)$	12.0296	14.4451	1.2145	2.6825	5.0549	8.5249	11.0920
$C(1\ldots6)$	12.0372	14.4521	1.1693	2.6700	5.0720	8.5415	11.1043
$C(1\ldots7)$	12.0293	14.4836	1.2552	2.6994	5.0060	8.5102	11.0342
$C(1\ldots8)$	12.0421	14.3329	1.1991	2.6797	5.2209	8.6499	11.3472
$C(124)$	12.0425	14.4601	1.2014	2.6793	5.0590	8.5378	11.1092
$C(1248)$	12.0337	14.4547	1.2028	2.6796	5.0577	8.5296	11.0952

Table 7-a: Results for new problems (Put options, $r > d$), I.

	Problem No. 1	2	3	4	5	6	7	8
S	70	70	70	70	70	70	70	70
r	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600
X	60	60	60	60	60	70	70	70
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	0.2500	0.5000	1.0000	2.0000	3.0000	0.2500	0.5000	1.0000
P_1	0.7099	1.6820	3.2003	5.1959	6.4529	3.9625	5.4320	7.2914
P_2	0.7100	1.6843	3.2182	5.2911	6.6749	3.9725	5.4629	7.3862
P_3	0.7106	1.6878	3.2319	5.3322	6.7498	3.9790	5.4776	7.4202
P_4	0.7110	1.6901	3.2400	5.3561	6.7916	3.9828	5.4862	7.4393
P_5	0.7113	1.6915	3.2451	5.3712	6.8183	3.9850	5.4915	7.4515
P_6	0.7116	1.6926	3.2487	5.3815	6.8366	3.9867	5.4951	7.4598
P_7	0.7119	1.6934	3.2514	5.3892	6.8498	3.9879	5.4979	7.4660
P_8	0.7120	1.6942	3.2533	5.3949	6.8599	3.9888	5.4999	7.4704
P_9	0.7120	1.6948	3.2545	5.4000	6.8672	3.9903	5.5023	7.4732
P_{10}	0.7129	1.6952	3.2569	5.4038	6.8738	3.9904	5.5021	7.4776
P_{11}	0.7123	1.6943	3.2577	5.4064	6.8803	3.9897	5.5034	7.4797
P_{12}	0.7126	1.6958	3.2579	5.4083	6.8837	3.9911	5.5050	7.4806
P_{13}	0.7127	1.6961	3.2595	5.4105	6.8870	3.9928	5.5045	7.4835
P_{14}	0.7124	1.6962	3.2599	5.4129	6.8908	3.9919	5.5061	7.4830
P_{15}	0.7131	1.6960	3.2613	5.4150	6.8937	3.9906	5.5067	7.4850
P_{16}	0.7124	1.6978	3.2620	5.4158	6.8947	3.9921	5.5059	7.4851
P_{17}	0.7129	1.6960	3.2617	5.4166	6.8976	3.9942	5.5072	7.4885
P_{18}	0.7123	1.6970	3.2615	5.4173	6.9002	3.9919	5.5056	7.4879
P_{19}	0.7129	1.6973	3.2621	5.4184	6.9027	3.9937	5.5076	7.4900
P_{20}	0.7132	1.6968	3.2623	5.4196	6.9028	3.9920	5.5093	7.4896
Exp. smoothed	0.7129	1.6971	3.2647	5.4209	6.9044	3.9926	5.5070	7.4903
$P(1\ldots4)$	0.7123	1.6987	3.2684	5.4450	6.9460	3.9963	5.5176	7.5096
$P(1\ldots5)$	0.7153	1.6970	3.2655	5.4340	6.9368	3.9909	5.5132	7.5051
$P(1\ldots6)$	0.7112	1.7024	3.2771	5.4328	6.9256	4.0113	5.5085	7.4930
$P(1\ldots7)$	0.7307	1.6998	3.2519	5.4580	6.9257	3.9510	5.5488	7.5615
$P(1\ldots8)$	0.6048	1.7313	3.2726	5.3698	6.9926	4.1034	5.4172	7.2535
$P(124)$	0.7125	1.6990	3.2703	5.4326	6.9122	3.9965	5.5146	7.4964
$P(1248)$	0.7134	1.6990	3.2680	5.4387	6.9381	3.9953	5.5152	7.5057

Table 7-b: Results for new problems (Put options, $r > d$), II.

	Problem No.						
	9	10	11	12	13	14	15
S	70	70	70	70	70	70	70
r	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600
X	70	70	80	80	80	80	80
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	2.0000	3.0000	0.2500	0.5000	1.0000	2.0000	3.0000
P_1	9.4225	10.6286	10.7075	11.7010	13.1405	14.8268	15.7097
P_2	9.7055	11.1528	10.7936	11.8478	13.4275	15.4459	16.7034
P_3	9.7881	11.2958	10.8152	11.8858	13.5022	15.6074	16.9665
P_4	9.8311	11.3658	10.8250	11.9035	13.5369	15.6803	17.0828
P_5	9.8583	11.4089	10.8307	11.9143	13.5576	15.7225	17.1489
P_6	9.8769	11.4385	10.8346	11.9214	13.5717	15.7508	17.1921
P_7	9.8905	11.4600	10.8374	11.9268	13.5819	15.7710	17.2231
P_8	9.9008	11.4765	10.8395	11.9307	13.5894	15.7864	17.2464
P_9	9.9078	11.4900	10.8409	11.9353	13.5952	15.7986	17.2646
P_{10}	9.9146	11.4996	10.8433	11.9355	13.5985	15.8075	17.2780
P_{11}	9.9201	11.5071	10.8436	11.9391	13.6046	15.8163	17.2926
P_{12}	9.9240	11.5142	10.8440	11.9408	13.6091	15.8228	17.3024
P_{13}	9.9289	11.5208	10.8447	11.9415	13.6083	15.8286	17.3096
P_{14}	9.9318	11.5250	10.8462	11.9427	13.6127	15.8340	17.3188
P_{15}	9.9357	11.5308	10.8463	11.9439	13.6166	15.8370	17.3234
P_{16}	9.9347	11.5337	10.8469	11.9450	13.6159	15.8395	17.3294
P_{17}	9.9393	11.5385	10.8489	11.9474	13.6197	15.8451	17.3332
P_{18}	9.9403	11.5401	10.8473	11.9450	13.6197	15.8453	17.3384
P_{19}	9.9424	11.5435	10.8494	11.9482	13.6214	15.8494	17.3433
P_{20}	9.9446	11.5481	10.8465	11.9471	13.6238	15.8523	17.3446
Exp. smoothed	9.9447	11.5473	10.8468	11.9454	13.6236	15.8522	17.3467
$P(1\dots4)$	9.9776	11.5819	10.8513	11.9545	13.6341	15.8683	17.3643
$P(1\dots5)$	9.9861	11.6083	10.8535	11.9625	13.6487	15.8959	17.4034
$P(1\dots6)$	9.9668	11.6050	10.8635	11.9554	13.6539	15.9158	17.4175
$P(1\dots7)$	9.9833	11.5675	10.8352	11.9967	13.6408	15.8555	17.4665
$P(1\dots8)$	9.9747	11.6793	10.8763	11.7976	13.5384	16.0719	17.2946
$P(124)$	9.9461	11.5460	10.8488	11.9475	13.6236	15.8645	17.3840
$P(1248)$	9.9793	11.5963	10.8537	11.9588	13.6429	15.8883	17.3937

Table 8-a: Results for new problems (Call options, $r < d$), I.

	Problem No. 1	2	3	4	5	6	7	8
S	70	70	70	70	70	70	70	70
r	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
X	60	60	60	60	60	70	70	70
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	0.2500	0.5000	1.0000	2.0000	3.0000	0.2500	0.5000	1.0000
C_1	10.4139	11.2076	12.4698	14.1189	15.1384	3.9823	5.4866	7.4387
C_2	10.4887	11.3249	12.6808	14.5460	15.8080	3.9901	5.5096	7.5070
C_3	10.5074	11.3552	12.7362	14.6583	15.9847	3.9957	5.5220	7.5346
C_4	10.5156	11.3692	12.7623	14.7100	16.0645	3.9989	5.5292	7.5505
C_5	10.5205	11.3776	12.7780	14.7405	16.1108	4.0009	5.5337	7.5605
C_6	10.5236	11.3834	12.7886	14.7611	16.1415	4.0024	5.5370	7.5676
C_7	10.5259	11.3874	12.7963	14.7762	16.1638	4.0035	5.5392	7.5727
C_8	10.5275	11.3906	12.8021	14.7874	16.1807	4.0043	5.5410	7.5762
C_9	10.5297	11.3928	12.8069	14.7964	16.1940	4.0052	5.5424	7.5800
C_{10}	10.5303	11.3945	12.8106	14.8034	16.2039	4.0059	5.5433	7.5823
C_{11}	10.5315	11.3971	12.8127	14.8100	16.2137	4.0061	5.5438	7.5833
C_{12}	10.5306	11.3974	12.8158	14.8145	16.2215	4.0072	5.5442	7.5858
C_{13}	10.5312	11.4004	12.8184	14.8193	16.2286	4.0057	5.5468	7.5853
C_{14}	10.5318	11.4004	12.8195	14.8217	16.2335	4.0057	5.5473	7.5875
C_{15}	10.5343	11.4013	12.8224	14.8253	16.2369	4.0057	5.5469	7.5893
C_{16}	10.5335	11.4012	12.8213	14.8274	16.2421	4.0076	5.5470	7.5903
C_{17}	10.5350	11.4022	12.8239	14.8305	16.2452	4.0072	5.5495	7.5917
C_{18}	10.5328	11.4020	12.8255	14.8326	16.2474	4.0078	5.5466	7.5914
C_{19}	10.5340	11.4049	12.8268	14.8345	16.2512	4.0078	5.5497	7.5908
C_{20}	10.5337	11.4022	12.8280	14.8366	16.2536	4.0060	5.5489	7.5926
Exp. smoothed	10.5329	11.4014	12.8231	14.8370	16.2551	4.0069	5.5471	7.5876
$C(1\ldots4)$	10.5356	11.4089	12.8374	14.8509	16.2701	4.0098	5.5548	7.6086
$C(1\ldots5)$	10.5429	11.4144	12.8483	14.8700	16.3013	4.0091	5.5515	7.6038
$C(1\ldots6)$	10.5310	11.4167	12.8415	14.8846	16.3020	4.0143	5.5709	7.6115
$C(1\ldots7)$	10.5791	11.3889	12.8582	14.8709	16.3558	4.0156	5.4840	7.5822
$C(1\ldots8)$	10.3581	11.5650	12.8055	14.7866	16.2023	3.9737	5.7557	7.5323
$C(124)$	10.5356	11.4041	12.8278	14.8408	16.2687	4.0110	5.5541	7.6002
$C(1248)$	10.5386	11.4125	12.8432	14.8649	16.2918	4.0103	5.5542	7.6053

Table 8-b: Results for new problems (Call options, $r < d$), II.

	Problem No.						
	9	10	11	12	13	14	15
S	70	70	70	70	70	70	70
r	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600	0.0600
X	70	70	80	80	80	80	80
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	2.0000	3.0000	0.2500	0.5000	1.0000	2.0000	3.0000
C_1	9.4225	10.6286	10.7075	11.7010	13.1405	14.8268	15.7097
C_2	9.7055	11.1528	10.7936	11.8478	13.4275	15.4459	16.7034
C_3	9.7881	11.2958	10.8152	11.8858	13.5022	15.6074	16.9665
C_4	9.8311	11.3658	10.8250	11.9035	13.5369	15.6803	17.0828
C_5	9.8583	11.4089	10.8307	11.9143	13.5576	15.7225	17.1489
C_6	9.8769	11.4385	10.8346	11.9214	13.5717	15.7508	17.1921
C_7	9.8905	11.4600	10.8374	11.9268	13.5819	15.7710	17.2231
C_8	9.9008	11.4765	10.8395	11.9307	13.5894	15.7864	17.2464
C_9	9.9078	11.4900	10.8409	11.9353	13.5952	15.7986	17.2646
C_{10}	9.9146	11.4996	10.8433	11.9355	13.5985	15.8075	17.2780
C_{11}	9.9201	11.5071	10.8436	11.9391	13.6046	15.8163	17.2926
C_{12}	9.9240	11.5142	10.8440	11.9408	13.6091	15.8228	17.3024
C_{13}	9.9289	11.5208	10.8447	11.9415	13.6083	15.8286	17.3096
C_{14}	9.9318	11.5250	10.8462	11.9427	13.6127	15.8340	17.3188
C_{15}	9.9357	11.5308	10.8463	11.9439	13.6166	15.8370	17.3234
C_{16}	9.9347	11.5337	10.8469	11.9450	13.6159	15.8395	17.3294
C_{17}	9.9393	11.5385	10.8489	11.9474	13.6197	15.8451	17.3332
C_{18}	9.9403	11.5401	10.8473	11.9450	13.6197	15.8453	17.3384
C_{19}	9.9424	11.5435	10.8494	11.9482	13.6214	15.8494	17.3433
C_{20}	9.9446	11.5481	10.8465	11.9471	13.6238	15.8523	17.3446
Exp. smoothed	9.9447	11.5473	10.8468	11.9454	13.6236	15.8522	17.3466
$C(1\dots4)$	9.9776	11.5819	10.8513	11.9545	13.6341	15.8683	17.3643
$C(1\dots5)$	9.9861	11.6083	10.8535	11.9625	13.6487	15.8959	17.4034
$C(1\dots6)$	9.9668	11.6050	10.8635	11.9554	13.6539	15.9158	17.4175
$C(1\dots7)$	9.9833	11.5675	10.8352	11.9967	13.6408	15.8555	17.4665
$C(1\dots8)$	9.9747	11.6793	10.8763	11.7976	13.5384	16.0719	17.2946
$C(124)$	9.9461	11.5460	10.8488	11.9475	13.6236	15.8645	17.3840
$C(1248)$	9.9793	11.5963	10.8537	11.9588	13.6429	15.8883	17.3937

Table 9-a: Results for new problems (Put options, $r < d$), I.

	Problem No.							
	1	2	3	4	5	6	7	8
S	70	70	70	70	70	70	70	70
r	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
X	60	60	60	60	60	70	70	70
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	0.2500	0.5000	1.0000	2.0000	3.0000	0.2500	0.5000	1.0000
P_1	0.8111	1.9967	4.0264	7.1482	9.5599	4.3297	6.1762	8.7974
P_2	0.8111	1.9967	4.0264	7.1486	9.5636	4.3297	6.1762	8.7974
P_3	0.8111	1.9967	4.0265	7.1494	9.5678	4.3297	6.1762	8.7976
P_4	0.8111	1.9968	4.0265	7.1497	9.5698	4.3297	6.1761	8.7976
P_5	0.8111	1.9969	4.0265	7.1500	9.5708	4.3297	6.1762	8.7977
P_6	0.8111	1.9970	4.0265	7.1502	9.5717	4.3297	6.1762	8.7976
P_7	0.8111	1.9963	4.0265	7.1503	9.5722	4.3297	6.1761	8.7977
P_8	0.8111	1.9962	4.0265	7.1505	9.5725	4.3297	6.1762	8.7977
P_9	0.8111	1.9972	4.0265	7.1507	9.5722	4.3297	6.1761	8.7976
P_{10}	0.8111	1.9966	4.0260	7.1510	9.5727	4.3297	6.1756	8.7975
P_{11}	0.8111	1.9974	4.0260	7.1505	9.5729	4.3297	6.1765	8.7979
P_{12}	0.8111	1.9959	4.0266	7.1504	9.5733	4.3298	6.1760	8.7979
P_{13}	0.8111	1.9978	4.0264	7.1511	9.5733	4.3298	6.1765	8.7976
P_{14}	0.8111	1.9981	4.0261	7.1507	9.5741	4.3298	6.1760	8.7970
P_{15}	0.8111	1.9965	4.0261	7.1504	9.5736	4.3298	6.1761	8.7972
P_{16}	0.8111	1.9964	4.0262	7.1503	9.5735	4.3298	6.1762	8.7979
P_{17}	0.8111	1.9982	4.0268	7.1504	9.5738	4.3298	6.1758	8.7965
P_{18}	0.8111	1.9960	4.0267	7.1513	9.5749	4.3298	6.1762	8.7975
P_{19}	0.8111	1.9964	4.0265	7.1513	9.5750	4.3298	6.1760	8.7979
P_{20}	0.8111	1.9964	4.0261	7.1512	9.5744	4.3298	6.1759	8.7978
Exp. smoothed	0.8111	1.9969	4.0264	7.1510	9.5739	4.3298	6.1761	8.7976
$P(1\ldots4)$	0.8111	1.9976	4.0266	7.1499	9.5731	4.3297	6.1750	8.7967
$P(1\ldots5)$	0.8111	1.9993	4.0255	7.1539	9.5733	4.3297	6.1804	8.8001
$P(1\ldots6)$	0.8111	1.9935	4.0286	7.1480	9.5946	4.3307	6.1656	8.7900
$P(1\ldots7)$	0.8111	1.8707	4.0249	7.1470	9.5034	4.3275	6.1952	8.8230
$P(1\ldots8)$	0.8111	2.5393	4.0284	7.1894	9.7474	4.3340	6.1558	8.7402
$P(124)$	0.8111	1.9969	4.0266	7.1515	9.5787	4.3297	6.1760	8.7979
$P(1248)$	0.8111	1.9950	4.0264	7.1513	9.5747	4.3298	6.1763	8.7978

Table 9-b: Results for new problems (Put options, $r < d$), II.

	Problem No.						
	9	10	11	12	13	14	15
S	70	70	70	70	70	70	70
r	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
X	70	70	80	80	80	80	80
σ	0.3	0.3	0.3	0.3	0.3	0.3	0.3
d	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400	0.0400
T	2.0000	3.0000	0.2500	0.5000	1.0000	2.0000	3.0000
P_1	12.4441	15.1250	11.3671	12.9070	15.3670	18.9695	21.6543
P_2	12.4465	15.1379	11.3671	12.9070	15.3673	18.9780	21.6878
P_3	12.4488	15.1466	11.3671	12.9069	15.3677	18.9829	21.7023
P_4	12.4499	15.1507	11.3672	12.9070	15.3680	18.9853	21.7096
P_5	12.4505	15.1530	11.3681	12.9070	15.3681	18.9866	21.7137
P_6	12.4510	15.1545	11.3677	12.9070	15.3682	18.9874	21.7164
P_7	12.4512	15.1556	11.3658	12.9069	15.3683	18.9882	21.7183
P_8	12.4514	15.1564	11.3652	12.9069	15.3683	18.9886	21.7197
P_9	12.4521	15.1568	11.3679	12.9068	15.3683	18.9883	21.7217
P_{10}	12.4516	15.1574	11.3667	12.9073	15.3681	18.9893	21.7221
P_{11}	12.4522	15.1583	11.3702	12.9071	15.3681	18.9907	21.7226
P_{12}	12.4521	15.1573	11.3645	12.9075	15.3682	18.9892	21.7228
P_{13}	12.4528	15.1583	11.3641	12.9072	15.3684	18.9893	21.7230
P_{14}	12.4526	15.1589	11.3709	12.9070	15.3678	18.9904	21.7239
P_{15}	12.4522	15.1591	11.3672	12.9070	15.3686	18.9900	21.7247
P_{16}	12.4528	15.1586	11.3677	12.9071	15.3686	18.9889	21.7247
P_{17}	12.4518	15.1601	11.3627	12.9070	15.3686	18.9906	21.7238
P_{18}	12.4522	15.1597	11.3671	12.9073	15.3675	18.9916	21.7259
P_{19}	12.4528	15.1600	11.3668	12.9074	15.3685	18.9898	21.7240
P_{20}	12.4527	15.1594	11.3668	12.9068	15.3677	18.9895	21.7241
Exp. smoothed	12.4524	15.1587	11.3669	12.9071	15.3682	18.9897	21.7233
$P(1\dots4)$	12.4519	15.1598	11.3684	12.9082	15.3690	18.9914	21.7298
$P(1\dots5)$	12.4523	15.1586	11.3872	12.9054	15.3682	18.9883	21.7265
$P(1\dots6)$	12.4587	15.1698	11.2913	12.9034	15.3668	18.9961	21.7301
$P(1\dots7)$	12.4281	15.1609	11.2145	12.9183	15.3825	19.0089	21.7218
$P(1\dots8)$	12.5006	15.1238	12.2916	12.9030	15.2985	18.9116	21.7822
$P(124)$	12.4549	15.1678	11.3674	12.9071	15.3690	18.9947	21.7347
$P(1248)$	12.4524	15.1609	11.3611	12.9068	15.3685	18.9913	21.7285

Table 10: Option values and their differences for Problem 3 of Table 3

C_1	15.7676									
C_2	15.9815	0.2139								
C_3	16.0434	0.0620	-0.1519							
C_4	16.0755	0.0320	-0.0299	0.1220						
C_5	16.0956	0.0202	-0.0119	0.0180	-0.1040					
C_6	16.1094	0.0138	-0.0064	0.0055	-0.0125	0.0915				
C_7	16.1195	0.0101	-0.0037	0.0027	-0.0028	0.0097	-0.0818			
C_8	16.1271	0.0076	-0.0025	0.0013	-0.0014	0.0014	-0.0083	0.0735		
C_9	16.1329	0.0058	-0.0018	0.0007	-0.0006	0.0008	-0.0006	0.0076	-0.0659	
C_{10}	16.1377	0.0048	-0.0011	0.0007	0.0000	0.0006	-0.0002	0.0004	-0.0072	0.0587
C_{11}	16.1415	0.0038	-0.0010	0.0001	-0.0006	-0.0006	-0.0012	-0.0010	-0.0014	0.0058
C_{12}	16.1448	0.0034	-0.0004	0.0005	0.0004	0.0010	0.0016	0.0028	0.0038	0.0052
C_{13}	16.1477	0.0029	-0.0004	0.0000	-0.0005	-0.0009	-0.0019	-0.0035	-0.0063	-0.0100
C_{14}	16.1500	0.0022	-0.0007	-0.0002	-0.0002	0.0003	0.0012	0.0031	0.0066	0.0128
C_{15}	16.1522	0.0022	0.0000	0.0007	0.0009	0.0011	0.0009	-0.0003	-0.0034	-0.0100
C_{16}	16.1539	0.0017	-0.0005	-0.0005	-0.0012	-0.0021	-0.0032	-0.0041	-0.0037	-0.0003
C_{17}	16.1552	0.0013	-0.0004	0.0001	0.0006	0.0017	0.0038	0.0070	0.0111	0.0148
C_{18}	16.1568	0.0016	0.0003	0.0007	0.0007	0.0001	-0.0016	-0.0054	-0.0124	-0.0235
C_{19}	16.1581	0.0013	-0.0003	-0.0006	-0.0013	-0.0020	-0.0021	-0.0005	0.0049	0.0173
C_{20}	16.1596	0.0015	0.0002	0.0005	0.0011	0.0025	0.0045	0.0066	0.0071	0.0022

Table 11: q–values, solutions of equations (3.5) and their differences for Problem 3 of Table 3

q_0	100.0000									
q_1	111.5089	11.5089								
q_2	116.7240	5.2151	-6.2938							
q_3	120.5273	3.8033	-1.4118	4.8820						
q_4	123.6842	3.1569	-0.6465	0.7653	-4.1167					
q_5	126.3721	2.6880	-0.4689	0.1776	-0.5877	3.5290				
q_6	128.7546	2.3824	-0.3055	0.1634	-0.0141	0.5736	-2.9554			
q_7	130.8913	2.1367	-0.2457	0.0598	-0.1036	-0.0895	-0.6630	2.2924		
q_8	132.8558	1.9645	-0.1723	0.0735	0.0137	0.1173	0.2068	0.8698	-1.4226	
q_9	134.6812	1.8254	-0.1391	0.0332	-0.0403	-0.0540	-0.1712	-0.3780	-1.2478	0.1748
q_{10}	136.4258	1.7446	-0.0808	0.0583	0.0252	0.0655	0.1194	0.2907	0.6687	1.9165
q_{11}	137.9055	1.4797	-0.2649	-0.1841	-0.2425	-0.2676	-0.3331	-0.4525	-0.7432	-1.4119
q_{12}	139.4035	1.4980	0.0182	0.2831	0.4673	0.7097	0.9774	1.3105	1.7630	2.5062
q_{13}	140.8710	1.4676	-0.0304	-0.0486	-0.3318	-0.7990	-1.5087	-2.4861	-3.7965	-5.5595
q_{14}	142.1866	1.3156	-0.1520	-0.1216	-0.0730	0.2588	1.0578	2.5665	5.0526	8.8492
q_{15}	143.4468	1.2602	-0.0554	0.0966	0.2182	0.2911	0.0323	-1.0255	-3.5920	-8.6446
q_{16}	144.7040	1.2572	-0.0030	0.0523	-0.0443	-0.2625	-0.5536	-0.5859	0.4395	4.0316
q_{17}	145.8227	1.1187	-0.1385	-0.1355	-0.1878	-0.1435	0.1189	0.6725	1.2584	0.8189
q_{18}	146.9801	1.1574	0.0388	0.1773	0.3127	0.5006	0.6441	0.5252	-0.1473	-1.4058
q_{19}	147.9962	1.0161	-0.1413	-0.1801	-0.3574	-0.6701	-1.1707	-1.8148	-2.3400	-2.1927

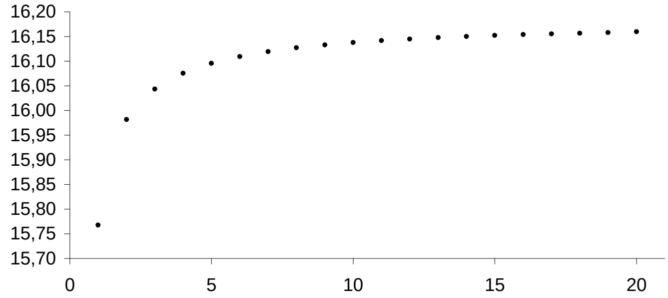

Figure 1: Option values for Problem 3 of Table 3

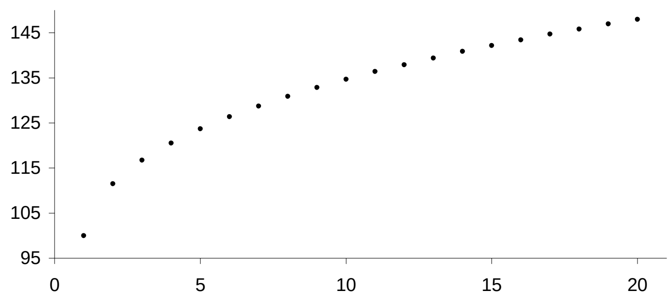

Figure 2: q – values for Problem 3 of Table 3

Table 12: Option values and their differences for Problem 24 of Table 1-c

P_1	1,1888									
P_2	1,1962	0,0074								
P_3	1,2023	0,0061	-0,0013							
P_4	1,2059	0,0036	-0,0025	-0,0012						
P_5	1,2083	0,0024	-0,0012	0,0013	0,0025					
P_6	1,2100	0,0017	-0,0007	0,0005	-0,0008	-0,0033				
P_7	1,2111	0,0011	-0,0006	0,0001	-0,0004	0,0004	0,0037			
P_8	1,2122	0,0011	0,0000	0,0006	0,0005	0,0009	0,0005	-0,0032		
P_9	1,2127	0,0005	-0,0006	-0,0006	-0,0012	-0,0017	-0,0026	-0,0031	0,0001	
P_{10}	1,2132	0,0005	0,0000	0,0006	0,0012	0,0024	0,0041	0,0067	0,0098	0,0097
P_{11}	1,2137	0,0005	0,0000	0,0000	-0,0006	-0,0018	-0,0042	-0,0083	-0,0150	-0,0248
P_{12}	1,2146	0,0009	0,0004	0,0004	0,0004	0,0010	0,0028	0,0070	0,0153	0,0303
P_{13}	1,2141	-0,0005	-0,0014	-0,0018	-0,0022	-0,0026	-0,0036	-0,0064	-0,0134	-0,0287
P_{14}	1,2149	0,0008	0,0013	0,0027	0,0045	0,0067	0,0093	0,0129	0,0193	0,0327
P_{15}	1,2151	0,0002	-0,0006	-0,0019	-0,0046	-0,0091	-0,0158	-0,0251	-0,0380	-0,0573
P_{16}	1,2153	0,0002	0,0000	0,0006	0,0025	0,0071	0,0162	0,0320	0,0571	0,0951
P_{17}	1,2156	0,0003	0,0001	0,0001	-0,0005	-0,0030	-0,0101	-0,0263	-0,0583	-0,1154
P_{18}	1,2147	-0,0009	-0,0012	-0,0013	-0,0014	-0,0009	0,0021	0,0122	0,0385	0,0968
P_{19}	1,2165	0,0018	0,0027	0,0039	0,0052	0,0066	0,0075	0,0054	-0,0068	-0,0453
P_{20}	1,2172	0,0007	-0,0011	-0,0038	-0,0077	-0,0129	-0,0195	-0,0270	-0,0324	-0,0256

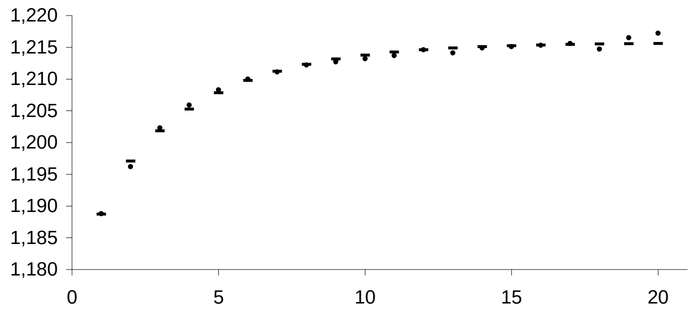

Figure 3: Option values (dots) and smoothed values according to the discrete exponential function with $m = 2, k = 1.2157, \alpha_1 = 0.0327, \beta_1 = 0,2854, \alpha_2 = 0.0348, \beta_2 = 2.6893$ (line segments) for Problem 24 of Table 1-c

9 Conclusions

We have formulated the problem of the numerical calculation of the Bermudan put and call options, with dividend, in the form of dynamic programming recursion. Based on that, a few properties of the Bermudan options have been derived. The asset price process is supposed to follow a geometric Brownian motion with drift and risk neutral valuation is used.

One of the main contributions of the paper are the closed form formulas for the values of the Bermudan put and call options. The Bermudan options are used as approximations of the corresponding American options when the time between now and the expiration is subdivided into n subintervals.

Probabilistic integrations are given to the formulas of the Bermudan options and it is shown that the option values converge to the American option values as $n \to \infty$.

We have also proposed numerical methods to obtain the values of the Bermudan put and call, by the use of several numerical integration methods, applied to the multivariate normal p.d.f. The results, obtained by the different methods showed very good agreements, thus we can rely on our numerical results. Among the multivariate normal integration techniques Genz's code turned out to be the fastest. If the individual integrals are computed with 5 digit accuracy then we claim 3 digit accuracy in the final results. However, another error may also come up, it is the probability with which the above-mentioned accuracy can in fact be obtained. In this respect the confidence level of 99% was used in connection with Genz's code.

The obtained sequences $\{P_i\}, \{C_i\}$ are smoothed by discrete, negative exponential functions of n, the number of subdividing intervals. The limiting values of the smoothing functions, as $n \to \infty$, are considered as the approximate values of the American options. Our approximate values are smaller, sometimes considerably smaller than those presented in the literature. For example, the binomial tree method systematically overestimates the option prices even if $n = 10,000$ or $n = 15,000$ steps are used. We have explained this phenomenon.

We have given closed form formulas for the Richardson extrapolation of the option prices. Two versions of them have been worked out: when all numbers of subdivisions between 1 and n are taken into account and when these numbers are powers of 2. The first version generalizes the formula of Geske and Johnson (1984), who applied Richardson extrapolation for the case of $n = 4$. However, we consider the Richardson extrapolation unreliable because it does not stabilize by the increase of the number of subdividing intervals.

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