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ON MULTIVARIATE DISCRETE MOMENT PROBLEMS AND THEIR APPLICATIONS TO BOUNDING EXPECTATIONS AND PROBABILITIES

Gergely Nagy   András Prékopa

Abstract. The discrete moment problem (DMP) has been formulated as a methodology to find the minimum and/or maximum of a linear functional acting on an unknown probability distribution, the support of which is a known discrete (usually finite) set, where some of the moments are known. The moments may be binomial, power or of more general type. The multivariate discrete moment problem (MDMP) has been initiated by the second named author who developed a linear programming theory and methodology for the solution of the DMP's and MDMP's under some assumptions, that concern the divided differences of the coefficients of the objective function. The central results in this respect are there that concern the structure of the dual feasible bases. In this paper further results are presented in connection with MDMP's for the case of power and binomial moments. The main theorem (Theorem 3.1) and its applications help us to find dual feasible bases under the assumption that the objective coefficient function has nonnegative divided differences of a given total order and further divided differences are nonnegative in each variable. Any dual feasible basis provides us with a bound for the discrete function that consists of the coefficients of the objective function and also for the linear functional. The latter bound is sharp if the basis is primal feasible as well. The combination of a dual feasible basis structure theorem and the dual method of linear programming is a powerful tool to find the sharp bound for the true value of the functional, i.e., the optimum value of the objective function. The lower and upper bounds are frequently close to each other even if the number of utilized moments is relatively small. Numerical examples are presented for bounding the expectations of functions of random vectors as well as probabilities of Boolean functions of event sequences.

Keywords: Discrete moment problem, Multivariate Lagrange interpolation, Linear programming, Expectation bounds, Probability bounds

1 Introduction

The multivariate discrete moment problem (MDMP) has been introduced and discussed in the papers by Prékopa (1992, 1998, 2000). The problem can be formulated in connection with a random vector $(X_1, \ldots, X_s)$ in the following way. We assume that the support of X_j is a known finite set $Z_j = \{z_{j0}, \ldots, z_{jn_j}\}$, where $z_{j0} < \cdots < z_{jn_j}$; $j = 1, \ldots, s$ and define

$$p_{i_1 \ldots i_s} = P(X_1 = z_{1i_1}, \ldots, X_s = z_{si_s}),\ 0 \le i_j \le n_j,\ j = 1, \ldots, s,$$

$$\mu_{\alpha_1 \ldots \alpha_s} = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s},$$

where $\alpha_1, \ldots, \alpha_s$ are nonnegative integers. The number $\mu_{\alpha_1, \ldots, \alpha_s}$ will be called the $(\alpha_1, \ldots, \alpha_s)$-order moment of the random vector $(X_1, \ldots, X_s)$, and the sum $\alpha_1 + \cdots + \alpha_s$ the total order of the moment.

Let $Z = Z_1 \times \cdots \times Z_s$ and $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ be a function for which we introduce some assumptions. Let $f_{i_1 \ldots i_s} = f(z_{1i_1}, \ldots, z_{si_s})$. One way to formulate the multivariate discrete moment problem is the following:

$$\begin{aligned} & \min(\max) \ \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \ldots i_s} p_{i_1 \ldots i_s} \\ \text{subject to} & \\ & \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s} = \mu_{\alpha_1 \ldots \alpha_s} \\ & \text{ for } \alpha_j \ge 0,\ j = 1, \ldots, s;\ \alpha_1 + \cdots \alpha_s \le m \\ & p_{i_1 \ldots i_s} \ge 0,\ \text{all } i_1, \ldots, i_s. \end{aligned} \tag{1}$$

We can generalize the above problem by introducing univariate moments of higher order than m into the constraints. One possible way, what we consider in this paper, is the following

$$\begin{aligned} & \min(\max) \ \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \ldots i_s} p_{i_1 \ldots i_s} \\ \text{subject to} & \\ & \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s} = \mu_{\alpha_1 \ldots \alpha_s} \\ & \text{ for } \alpha_j \ge 0,\ j = 1, \ldots, s;\ \alpha_1 + \cdots \alpha_s \le m \text{ and} \\ & \text{ for } \alpha_j = 0,\ j = 1, \ldots, k-1, k+1, \ldots, s,\ m \le \alpha_k \le m_k,\ k = 1, \ldots, s; \\ & p_{i_1 \ldots i_s} \ge 0,\ \text{all } i_1, \ldots, i_s. \end{aligned} \tag{2}$$

In problems (1) and (2) the unknown variables are the $p_{i_1 \ldots i_s}$, all other quantities are known. In case of (2), this means that, in addition to all moments of total order at most m, the at most $m_k th$ order moments $(m_k \ge m)$ of the kth univariate marginal distribution is also known, $k = 1, \ldots, s$.

The above problems serve for bounding

$$E[f(X_1,\ldots,X_s)] \tag{3}$$

under the given moment information. By suitable choices of the function f, the expectation (3) specializes to

$$P(X_1 \geq r_1,\ldots,X_s \geq r_s) \tag{4}$$

or

$$P(X_1 = r_1,\ldots,X_s = r_s), \tag{5}$$

where $(r_1,\ldots,r_s) \in Z$. As byproducts of our methodology, we also obtain bounds for the discrete function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$.

Problems (1) and (2) can be written in more compact forms by the use of the tensor products of matrices. The tensor product $B \otimes C$ of the $m_1 \times n_1$ matrix $B = (b_{ij})$ and the $m_2 \times n_2$ matrix $C = (c_{ij})$ is the $m_1 m_2 \times n_1 n_2$ matrix $B \otimes C = (c_{ij}B)$. It is well-known (see, e.g., Horn and Johnson (1991)) that the tensor product is associative but not commutative. Let us introduce the notations:

$$A_j = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_{j0} & z_{j1} & \cdots & z_{jn_j} \\ \vdots & & \ddots & \\ z_{j0}^{m_j} & z_{j1}^{m_j} & \cdots & z_{jn_j}^{m_j} \end{pmatrix},$$

$$A = A_1 \otimes \cdots \otimes A_s,$$

$$\begin{aligned} \boldsymbol{b} &= E\left[(1,X_1,\ldots,X_1^{m_1}) \otimes \cdots \otimes (1,X_s,\ldots,X_s^{m_s})\right]^T \\ &= (\mu_{00\ldots0},\mu_{10\ldots0},\ldots,\mu_{m_10\ldots0},\mu_{010\ldots0},\mu_{11\ldots0},\ldots)^T \\ \boldsymbol{p} &= (p_{i_1\ldots i_s},\ 0 \leq i_1 \leq n_1,\ldots,0 \leq i_s \leq n_s)^T \\ \boldsymbol{f} &= (f_{i_1\ldots i_s},\ 0 \leq i_1 \leq n_1,\ldots,0 \leq i_s \leq n_s)^T, \end{aligned}$$

where the ordering of the components in $\boldsymbol{p}$ and $\boldsymbol{f}$ coincides with that of the corresponding columns in the matrix A. By the aid of suitable selections of the rows of A, as well as components of $\boldsymbol{b}$, we can write up the above problems in compact forms. The compact form of problem (1) is written as:

$$\begin{array}{ll} & \min(\max) \quad \boldsymbol{f}^T\boldsymbol{p} \\ \text{subject to} & \\ & \begin{aligned} \tilde{A}\boldsymbol{p} &= \tilde{\boldsymbol{b}} \\ \boldsymbol{p} &\geq \mathbf{0}, \end{aligned} \end{array} \tag{6}$$

and the compact form of problem (2) is:

$$\begin{array}{ll} & \min(\max) \quad \boldsymbol{f}^T\boldsymbol{p} \\ \text{subject to} & \\ & \begin{aligned} \hat{A}\boldsymbol{p} &= \hat{\boldsymbol{b}} \\ \boldsymbol{p} &\geq \mathbf{0}. \end{aligned} \end{array} \tag{7}$$

The matrix A has size $[(m_1+1)\cdots(m_s+1)]\times[(n_1+1)\cdots(n_s+1)]$ while $\widetilde{A}$ has size $N\times[(n_1+1)\cdots(n_s+1)]$, where $N=\binom{s+m}{m}$, and $\widehat{A}$ has size $N'\times[(n_1+1)\cdots(n_s+1)]$, where $N'=N+\sum_{j=1}^{s}(m_j-m)$. The matrix $\widetilde{A}$ has full rank if $m\le n_j,\ j=1,\ldots,s$ and $\widehat{A}$ has full rank if $m_j\le n_j,\ j=1,\ldots,s$.

Let V_{min} (V_{max}) designate the minimum (maximum) value in problem (1) or problem (2). Let further B_1 (B_2) designate a dual feasible basis (i.e., a basis for which the optimality condition is satisfied) for the minimization (maximization) problem. Then, by linear programming theory, we know that

$$\boldsymbol{f}_{B_1}^T\boldsymbol{p}_{B_1}\le V_{min}\le E\left[f(X_1,\ldots,X_s)\right]\le V_{max}\le \boldsymbol{f}_{B_2}^T\boldsymbol{p}_{B_2}. \tag{8}$$

If B_1 (B_2) is an optimal basis in the minimization (maximization) problem, then the first (last) inequality holds with equality sign. We say that V_{min} and V_{max} are the sharp lower and upper bounds, respectively, for the expectation of $f(X_1,\ldots,X_s)$.

The formulation of the discrete binomial moment problem is similar to the discrete power moment problem. Taking into account its most important applications to event sequences, where X_j means the number of events that occur in the jth sequence, $j=1,\ldots,s$, we formulate the problem for the case of $Z_j=\{0,\ldots,n_j\},\ j=1,\ldots,s$.

Let us introduce the cross binomial moments of order $(\alpha_1,\ldots,\alpha_s)$ ($\alpha_1,\ldots,\alpha_s$ are non-negative integers):

$$S_{\alpha_1\ldots\alpha_s}=E\left[\binom{X_1}{\alpha_1}\cdots\binom{X_s}{\alpha_s}\right] \tag{9}$$

and formulate again two different types of problems. The first one is

$$\begin{aligned} &\min(\max)\quad \sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s} f_{i_1\ldots i_s}p_{i_1\ldots i_s}\\ \text{subject to}\quad &\\ &\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}\binom{i_1}{\alpha_i}\cdots\binom{i_s}{\alpha_s}p_{i_1\ldots i_s}=S_{\alpha_1\ldots\alpha_s}\\ &\text{for}\quad \alpha_j\ge 0,\quad j=1,\ldots,s,\quad \alpha_1+\cdots+\alpha_s\le m\\ &p_{i_1\ldots i_s}\ge 0,\quad \text{all } i_1,\ldots,i_s, \end{aligned} \tag{10}$$

while the second one is

$$\begin{aligned} &\min(\max)\quad \sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s} f_{i_1\ldots i_s}p_{i_1\ldots i_s}\\ \text{subject to}\quad &\\ &\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}\binom{i_1}{\alpha_i}\cdots\binom{i_s}{\alpha_s}p_{i_1\ldots i_s}=S_{\alpha_1\ldots\alpha_s}\\ &\text{for } \alpha_j\ge 0,\ j=1,\ldots,s;\ \alpha_1+\cdots\alpha_s\le m \text{ and}\\ &\text{for } \alpha_j=0,\ j=1,\ldots,k-1,k+1,\ldots,s,\ m\le\alpha_k\le m_k,\ k=1,\ldots,s;\\ &p_{i_1\ldots i_s}\ge 0,\ \text{all } i_1,\ldots,i_s. \end{aligned} \tag{11}$$

These correspond to problems (1) and (2), respectively. If in problems (1) and (2) we assume that $Z_j = \{0, \ldots, n_j\}$, $j = 1, \ldots, s$, then problems (1) and (10) as well as problems (2) and (11) can be transformed into the each other by simple nonsingular transformations. This means that if we write up problem (10) (problem (11)) in the compact matrix form of (6) ((7)), then the matrices of the equality constraints can be transformed into each other by a nonsingular square matrix and its inverse, respectively. This fact implies that a basis in problem (6) (problem (7)) is dual feasible if and only if it is dual feasible in problem (10) (problem (11)). In fact, let D designate the nonsingular square matrix that has the property that DA equals the matrix of the equality constraints in problem (10). Then the optimality condition for a basis B in problem (6) is:

$$\boldsymbol{f}_B^T B^{-1} \boldsymbol{a}_k \leq (\geq) f_k \text{ for all } k, \tag{12}$$

while the optimality condition for the transformed basis in problem (11) is:

$$\boldsymbol{f}_B^T (DB)^{-1} D \boldsymbol{a}_k \leq (\geq) f_k \text{ for all } k. \tag{13}$$

Obviously, (12) and (13) are the same. The above reasoning applies to problems (7) and (11) as well.

Finally, in order to see the relationship between multivariate Lagrange interpolation and dual feasible bases of problems (6) ((7)), let $U = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_M\}$ be a set of distinct points in $\mathbb{R}^s$ and $H = \{(\alpha_1, \ldots, \alpha_s)\}$ a finite set of s-tuples of nonnegative integers $(\alpha_1, \ldots, \alpha_s)$.

We say that the set U admits an H-type Lagrange interpolation if for any real function $f(\boldsymbol{z})$, $\boldsymbol{z} \in U$, there exists a polynomial $p(\boldsymbol{z})$ of the form

$$p(\boldsymbol{z}) = \sum_{(\alpha_1, \ldots, \alpha_s) \in H} c(\alpha_1, \ldots, \alpha_s) z_1^{\alpha_1} \cdots z_s^{\alpha_s}, \tag{14}$$

where all $c(\alpha_1, \ldots, \alpha_s)$ are real, such that

$$p(\boldsymbol{u}_i) = f(\boldsymbol{u}_i), \ i = 1, \ldots, M. \tag{15}$$

Let us define $\widetilde{\boldsymbol{b}}(z_1, \ldots, z_s)$ $(\widehat{\boldsymbol{b}}(z_1, \ldots, z_s))$ in a similar way as we have defined $\widetilde{\boldsymbol{b}}$ $(\widehat{\boldsymbol{b}})$ but we remove the expectation and replace z_j for X_j, $j = 1, \ldots, s$.

In connection with problem (6) (problem (7)) we define H, I and U as follows:

$$\begin{array}{rcl} H & = & \{(\alpha_1, \ldots, \alpha_s) |\ 0 \leq \alpha_j,\ \alpha_j \text{ integer},\ \alpha_1 + \cdots + \alpha_s \leq m,\ j = 1, \ldots, s\} \\ (H & = & \{(\alpha_1, \ldots, \alpha_s) |\ 0 \leq \alpha_j,\ \alpha_j \text{ integer},\ \alpha_1 + \cdots + \alpha_s \leq m,\ j = 1, \ldots, s; \\ & & \text{or } \alpha_j = 0,\ j = 1, \ldots, k-1, k+1, \ldots, s,\ m \leq \alpha_k \leq m_k,\ k = 1, \ldots, s\}), \end{array} \tag{16}$$

$$\begin{array}{rcl} I & = & \{(i_1, \ldots, i_s) |\ \widetilde{a}_{i_1 \cdots i_s} \in \widetilde{B}\} \\ (I & = & \{(i_1, \ldots, i_s) |\ \widehat{a}_{i_1 \cdots i_s} \in \widehat{B}\}), \end{array}$$

$$U = \{(z_{1i_1}, \ldots, z_{si_s}) |\ (i_1, \ldots, i_s) \in I\}.$$

Then

$$\begin{array}{rcl} L_I(z_1,\ldots,z_s) & = & \boldsymbol{f}_{\widetilde{B}}^T \widetilde{B}^{-1}\widetilde{\boldsymbol{b}}(z_1,\ldots,z_s) \\ (L_I(z_1,\ldots,z_s) & = & \boldsymbol{f}_{\widehat{B}}^T \widehat{B}^{-1}\widehat{\boldsymbol{b}}(z_1,\ldots,z_s)) \end{array} \tag{17}$$

is the unique H-type Lagrange polynomial corresponding to the set U.

The dual feasibility of the basis $\widetilde{B}$ or $\widehat{B}$ in the minimization (maximization) problem means that

$$\begin{array}{l} f(z_1,\ldots,z_s) \geq L_I(z_1,\ldots,z_s),\ \text{all } (z_1,\ldots,z_s) \in Z \\ (f(z_1,\ldots,z_s) \leq L_I(z_1,\ldots,z_s),\ \text{all } (z_1,\ldots,z_s) \in Z), \end{array} \tag{18}$$

where equality holds in case of $(z_1,\ldots,z_s) \in U$.

Relation (18) is called the condition of optimality of the minimization (maximization) problem (6), (7).

Replacing $(X_1,\ldots,X_s)$ for $(z_1,\ldots,z_s)$ and taking expectations in (18) we obtain bounds for $E[f(X_1,\ldots,X_s)]$. If the basis is also primal feasible, then it is optimal and thus, the obtained bound is sharp.

The organization of the paper is the following. In Section 2 we introduce the concept of a discrete convex function in the multivariate case and mention some of its properties. In Section 3 we prove a theorem on multivariate Lagrange interpolation that generalizes the well known univariate formula for the difference of the function and the interpolating polynomial and also the main theorem (Theorem 4.1) in Prékopa (1998). Assuming that the coefficient function in problem (2) satisfies some higher order convexity conditions, we present some bounds in Section 4. In Section 5 we introduce algorithms to generate a variety of dual feasible bases in the bivariate case. Finally, in Section 6, numerical examples are presented.

2 Multivariate Discrete Higher Order Convex Functions

Let $f(z),\ z \in \{z_0,\ldots,z_n\}$ be a univariate discrete function, where $z_0 < \cdots < z_n$. Its first order divided differences are designated and defined by the equation

$$[z_{i_1}, z_{i_2}; f] = \frac{f(z_{i_1}) - f_(z_{i_2})}{z_{i_1} - z_{i_2}}. \tag{19}$$

The kth order divided differences are defined by induction in the usual way (see Jordan (1965), Popoviciu (1944), Prékopa (1998)).

We call the function kth order convex if its kth order divided differences are all nonnegative. First order convexity means monotonicity, second order convexity means convexity of the sequence of function values in the traditional sense.

Note that this definition is slightly different than that given by Popoviciu (1944). In Popoviciu (1944) the function f is called kth order convex if its $k+1st$ order divided differences are nonnegative.

If we consider a multivariate discrete function $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z = Z_1 \times \cdots \times Z_s$ (see Section 1) and take the subset

$$\begin{aligned} Z_{I_1 \dots I_s} &= \{z_{1i},\ i \in I_1\} \times \cdots \times \{z_{si},\ i \in I_s\} \\ &= Z_{1I_1} \times \cdots \times Z_{sI_s}, \end{aligned} \tag{20}$$

where $|I_j| = k_j + 1,\ j = 1, \dots, s$, then we can define the $(k_1, \dots, k_s)$-order divided difference of f on the set (20) in an iterative way. First we take the $k_1 th$ divided difference with respect to the first variable, then the $k_2 th$ divided difference with respect to the second variable etc.. This operations can be executed in any order even in a mixed manner, the result is always the same. Let

$$[z_{1i},\ i \in I_1; \cdots; z_{si},\ i \in I_s; f] \tag{21}$$

designate the $(k_1, \dots, k_s)$-order divided difference. The sum $k_1 + \cdots + k_s$ is called the total order of the divided difference.

The above mentioned statement concerning divided differences is essentially the same as the following two statements.

If $f(z),\ z \in Z$ is a univariate discrete function and $V_1, V_2 \in Z,\ V_1 \cap V_2 = \emptyset$, then

$$[V_1; [V_2; f]] = [V_2; [V_1; f]] = [V_1 \cup V_2; f].$$

If $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z = Z_1 \times Z_2$ is a discrete function and $z_1 \in Z_1,\ z_2 \in Z_2,\ V_1 \subset Z_1,\ V_2 \subset Z_2$, then

$$[V_1; [z_1; V_2; f]] = [V_2; [V_1; z_2; f]] = [V_1; V_2; f].$$

Definition 2.1 *The function $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ is called a (multivariate) discrete convex function of order $(m_1, \dots, m_s)$ if for any $\{z_{ji}, i \in I_j\}$, $|I_j| = m_j + 1,\ j = 1, \dots, s$ we have the relation*

$$[z_{1i}, i \in I_1; \cdots; z_{si}, i \in I_s; f] \geq 0. \tag{22}$$

Definition 2.2 *The function $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ is called a (multivariate) discrete convex function of order m, if all its divided differences of total order m are nonnegative.*

If $f(\boldsymbol{z}),\ g(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ are convex of the same order, then this property carries over to the sum $f(\boldsymbol{z}) + g(\boldsymbol{z}),\ \boldsymbol{z} \in Z$. As regards the product, we have the following

Theorem 2.1 *If $f(\boldsymbol{z}) \geq 0,\ g(\boldsymbol{z}) \geq 0,\ \boldsymbol{z} \in Z$ are convex of any order $i,\ 1 \leq i \leq m$, then the same holds for the function $f(\boldsymbol{z})g(\boldsymbol{z}),\ \boldsymbol{z} \in Z$.*

Proof. The divided differences of a product can be obtained by a rule similar to the derivatives of a product. The assertion easily follows from this fact. □

Our definitions of higher order convexity use only divided differences in the directions of the coordinate axes.

It may happen, e.g., that a function has all nonnegative second total order divided differences but it does not produce a convex discrete function along a line. An example is given below.

Let $Z_1 = Z_2 = \{0,1,2\}$ and define $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z_1 \times Z_2$ in the following way:

$$\begin{array}{lll} f(0,0) = 0, & f(1,0) = 1.2, & f(2,0) = 2.6, \\ f(0,1) = 0.4, & f(1,1) = 2, & f(2,1) = 3.6, \\ f(0,2) = 1, & f(1,2) = 2.8, & f(2,2) = 4.6. \end{array}$$

The function is not convex along the line $(0,2),\ (1,1),\ (2,0)$. In fact, we have

$$f(1,1) = 2 > \frac{1+2.6}{2} = \frac{f(0,2)+f(2,0)}{2}.$$

As we see in later sections of the paper, we are able to derive quite good bounds based on our more restrictive definition of multivariate discrete convex functions. However, the inclusion of the condition of nonnegativity of the divided differences along any set of orthogonal directions would improve on the results.

If $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ is derived from a function $\overline{f}(\boldsymbol{z})$ defined in $\overline{Z} = [z_{10}, z_{1n_1}] \times \cdots \times [z_{s0}, z_{sn_s}]$ by taking $f(\boldsymbol{z}) = \overline{f}(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ and $\overline{f}(\boldsymbol{z})$ has continuous, nonnegative derivatives of order $(k_1,\ldots,k_s)$ in the interior of $\overline{Z}$, then all divided differences of $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ of order $(k_1,\ldots,k_s)$ are nonnegative. For further results in this respect see Popoviciu (1944).

Given a function $f(\boldsymbol{z}),\ \boldsymbol{z} \in Z$ which is discrete convex of order m, it is a difficult task to construct an $\overline{f}(\boldsymbol{z}),\ \boldsymbol{z} \in \overline{Z}$ with continuous, nonnegative derivatives of total order m. We can easily do it, however, if we restrict the definition of f to a subset of Z. One of such constructions is expressed by

Theorem 2.2 *Define the simplicial discrete set Z_I in the following way:*

$$Z_I = \{(z_{i_1},\ldots,z_{i_s}) |\ (i_1,\ldots,i_s) \in I\}, \tag{23}$$

where

$$I = \{(i_1,\ldots,i_s) |\ i_1 + \cdots + i_s \le m,\ 0 \le i_j \le n_j,\ j = 1,\ldots,s\}, \tag{24}$$
$$m \le n_1 + \cdots + n_s.$$

Then there exists a unique polynomial $L_I(\boldsymbol{z})$ such that

$$L_I(\boldsymbol{z}) = f(\boldsymbol{z}) \text{ for } \boldsymbol{z} \in Z_I$$

and the $(k_1,\ldots,k_s)$-order derivative of $L_I(\boldsymbol{z})$ is equal to the $(k_1,\ldots,k_s)$-order divided difference of the function f corresponding to the set

$$\{(z_{1i_1},\ldots,z_{si_s}) |\ 0 \le i_j \le k_j,\ j = 1,\ldots,s\}.$$

The polynomial $L_I(\boldsymbol{z})$ is given by

$$L_I(z_1,\ldots,z_s) = \sum_{\substack{i_1+\cdots+i_s \le m \\ 0 \le i_j \le n_j,\ j=1,\ldots,s}} [z_{10},\ldots,z_{1i_1};\cdots;z_{s0},\ldots,z_{si_s};f] \prod_{j=1}^{s} \prod_{h=0}^{i_j-1} (z_j - z_{jh}), \tag{25}$$

where, by definition, $\displaystyle\prod_{h=1}^{i_j-1} (z_j - z_{jh}) = 1$, *for* $i_j = 0$.

Proof. It is easy to check that $L_I(\boldsymbol{z})$ has the required derivatives. The unicity of the polynomial is proved in Prékopa (1998, p.362, Theorem 4.1). □

Remark 2.1 *The polynomial $L_I(\boldsymbol{z})$ is the Newton's form of the multivariate Lagrange polynomial corresponding to the set of points Z_I.*

In Prékopa (1998) bounds for $E[f(X_1,\ldots,X_s)]$ are presented by the use of expectations of Lagrange polynomials of $X_1,\ldots,X_s$ for the case where the moments $\mu_{\alpha_1\ldots\alpha_s}$ are known for $\alpha_1+\cdots\alpha_s\le m$ and the function $f(\boldsymbol{z}),\ \boldsymbol{z}\in Z$ satisfies some higher order convexity requirement, e.g., it is a discrete convex function of order $m+1$. Simultaneously, bounds are presented for the function $f(\boldsymbol{z}),\ \boldsymbol{z}\in Z$ itself. We use this technique in this paper for more general problems.

We will frequently use the following formula, well-known in univariate Lagrange interpolation theory:

$$f(z)-L(z)=[z_0,\ldots,z_k,z;f]\prod_{j=0}^{k}(z-z_j), \tag{26}$$

where $L(z)$ is the Lagrange polynomial corresponding to the base points $z_0,\ldots,z_k$, i.e.,

$$L(z)=\sum_{i=0}^{k}f(z_i)\frac{(z-z_0)\cdots(z-z_{i-1})(z-z_{i+1})\cdots(z-z_k)}{(z_i-z_0)\cdots(z_i-z_{i-1})(z_i-z_{i+1})\cdots(z_i-z_k)}. \tag{27}$$

Formula (26) has been established for functions defined in an interval. However, we will use it in connection with discrete functions, where not only the set of base points is a finite set but also the whole set on which f is defined.

3 A Theorem on Multivariate Lagrange Interpolation

In this section we drop the condition that $Z_1,\ldots,Z_s$ are ordered sets and prove a theorem valid for a Lagrange interpolation polynomial defined in $\mathbb{R}^s$. We consider the set of subscripts

$$I=I_0\cup\left(\cup_{j=1}^{s}I_j\right), \tag{28}$$

where

$$I_0=\{(i_1,\ldots,i_s)|\ 0\le i_j\le m-1,\ \text{integers},\ j=1,\ldots,s,\ i_1+\ldots+i_s\le m\} \tag{29}$$

and

$$\begin{aligned} I_j&=\{(i_1,\ldots,i_s)|\ i_j\in K_j,\ i_l=0\ l\ne j\}\\ K_j&=\{k_j^{(1)},\ldots,k_j^{(|K_j|)}\}\subset\{m,m+1,\ldots,n_j\},\ j=1,\ldots,s.\end{aligned} \tag{30}$$

In what follows we will use the notations

$$\begin{aligned} Z_{ji}&=\{z_{j0},\ldots,z_{ji}\},\\ Z'_{ji}&=\{z_{j0},\ldots,z_{ji},z_j\},\\ &\quad i=0,\ldots,n_j,\ j=1,\ldots,s\end{aligned}$$

and

$$\begin{aligned} K_{ji} &= \{k_j^{(1)}, \dots, k_j^{(i)}\}, \\ Z_{jK_{ji}} &= \{z_{jk_j^{(1)}}, \dots, z_{jk_j^{(i)}}\}, \\ & \quad i = 1, \dots, |K_j|, \; j = 1, \dots, s, \\ Z_{jK_j} &= Z_{jK_{j|K_j|}}, \; j = 1, \dots, s. \end{aligned}$$

Corresponding to the points $Z_I = \{(z_{1i_1}, \dots, z_{si_s}) | \, (i_1, \dots, i_s) \in I\}$ we assign the Lagrange polynomial, given by its Newton's form

$$\begin{aligned} & L_I(z_1, \dots, z_s) \\ = & \sum_{\substack{i_1+\dots+i_s \le m \\ 0 \le i_j \le m-1, \; j=1,\dots,s}} [Z_{1i_1}; \cdots; Z_{si_s}; f] \prod_{j=1}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \\ & + \sum_{j=1}^{s} \sum_{i=1}^{|K_j|} \left[Z_{10}; \cdots; Z_{(j-1)0}; Z_{j(m-1)} \cup Z_{jK_{ji}}; Z_{(j+1)0}; \cdots; Z_{s0}; f\right] \prod_{k \in \{0,\dots,m-1\} \cup K_{j(i-1)}} (z_j - z_{jk}), \\ & \text{where, by definition, } \prod_{k=0}^{i_j-1} (z_j - z_{jk}) = 1, \text{ for } i_j = 0, \text{ and } K_{j0} = \emptyset. \end{aligned} \tag{31}$$

In (31) the function f is not necessarily restricted to the set Z as its domain of definition; it may be defined on any subset of $\mathbb{R}^s$ that contains Z.

Next, we define the "residual function":

$$R_I(z_1, \dots, z_s) = R_{1I}(z_1, \dots, z_s) + R_{2I}(z_1, \dots, z_s), \tag{32}$$

where

$$\begin{aligned} & R_{1I}(z_1, \dots, z_s) \\ = & \sum_{j=1}^{s} \left[z_{10}; \cdots; z_{(j-1)0}; Z_{j(m-1)} \cup Z_{jK_j} \cup \{z_j\}; z_{(j+1)0}; \cdots; z_{s0}; f\right] \prod_{k \in \{0,\dots,m-1\} \cup K_j} (z_j - z_{jk}) \end{aligned} \tag{33}$$

and

$$\begin{aligned} & R_{2I}(z_1, \dots, z_s) \\ = & \sum_{h=1}^{s} \sum_{\substack{i_h+\cdots+i_s=m \\ 0 \le i_j \le m-1, \; j=h,\dots,s}} \left[z_1; \cdots; z_{h-1}; Z'_{hi_h}; Z_{(h+1)i_{h+1}}; \cdots; Z_{si_s}; f\right] \prod_{l=0}^{i_h} (z_h - z_{hl}) \\ & \times \prod_{h+1}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \\ & + \sum_{j=h+1}^{s} \left[z_1; \cdots; z_{h-1}; Z'_{h0}; Z_{(h+1)0}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}\right] (z_h - z_{h0}) \\ & \times \prod_{k=0}^{m-1} (z_j - z_{jk}). \end{aligned} \tag{34}$$

The following theorem generalizes the univariate formula (26) and the multivariate formula in Prékopa (1998, Section 4).

Theorem 3.1 *Consider the Lagrange polynomial (31), corresponding to the points Z_I. For any $\boldsymbol{z} = (z_1, \ldots, z_s)$ for which the function f is defined, we have the equality*

$$L_I(z_1, \ldots, z_s) + R_I(z_1, \ldots, z_s) = f(z_1, \ldots, z_s). \tag{35}$$

Proof. For the sake of simplicity we assume that $m_j \le n_j$, $j = 1, \ldots, s$. The proof of the general case needs only slight modification. We may assume, without loss of generality, that $K_j = \{m, m+1, \ldots, m_j\}$, where $m_j \ge m$, $j = 1, \ldots, s$. In fact, if we introduce the new sets $\overline{Z}_j = Z_{j(m-1)} \cup Z_{jK_j}$, $j = 1, \ldots, s$, and prove the assertion for them, we will have proved the statement for the general case.

Under the assumption for the sets K_j, $j = 1, \ldots, s$, the functions $L_I(z_1, \ldots, z_s)$ and $R_{1I}(z_1, \ldots, z_s)$ specialize as follows:

$$\begin{aligned} &L_I(z_1, \ldots, z_s) \\ &= \sum_{\substack{i_1+\ldots+i_s \le m \\ 0 \le i_j \le m-1,\ j=1,\ldots,s}} [Z_{1i_1}; \cdots; Z_{si_s}; f] \prod_{j=1}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \\ &+ \sum_{j=1}^{s} \sum_{i_j=m}^{m_j} \left[Z_{10}; \cdots; Z_{(j-1)0}; Z_{ji_j}; Z_{(j+1)0}; \cdots; Z_{s0}; f\right] \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \end{aligned} \tag{36}$$

and

$$R_{1I}(z_1, \ldots, z_s) = \sum_{j=1}^{s} \left[Z_{10}; \cdots; Z_{(j-1)0}; Z'_{jm_j}; Z_{(j+1)0}; \cdots; Z_{s0}; f\right] \prod_{k=0}^{m_j} (z_j - z_{jk}). \tag{37}$$

The formula for $R_{2I}(z_1, \ldots, z_s)$ remains unchanged. Now we prove the following

Lemma 3.2 *We have the equality*

$$\begin{aligned} &L_I(z_1, \ldots, z_s) + R_{1I}(z_1, \ldots, z_s) \\ &= \sum_{\substack{i_1+\cdots+i_s \le m \\ 0 \le i_j \le m-1,\ j=1,\ldots,s}} [Z_{1i_1}; \cdots; Z_{si_s}; f] \prod_{j=1}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \\ &+ \sum_{j=1}^{s} \left[Z_{10}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}; f\right] \prod_{k=0}^{m-1} (z_j - z_{jk}). \end{aligned} \tag{38}$$

Proof of Lemma 3.2 Consider the function of the single variable z_j:

$$[Z_{10}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}; f].$$

Its Lagrange polynomial, corresponding to the points $z_{jm}, \dots, z_{jm_j}$, equals

$$\sum_{i_j=m}^{m_j} [Z_{10}; \cdots; Z_{(j-1)0}; Z_{ji_j}; Z_{(j+1)0}; \cdots; Z_{s0}; f] \prod_{k=m}^{i_j-1} (z_j - z_{jk}).$$

Hence, by formula (26), we have the equation

$$\begin{aligned} &[Z_{10}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}; f] \\ &= \sum_{i_j=m}^{m_j} [Z_{10}; \cdots; Z_{(j-1)0}; Z_{ji_j}; Z_{(j+1)0}; \cdots; Z_{s0}; f] \prod_{k=m}^{i_j-1} (z_j - z_{jk}) \\ &+ \left[Z_{10}; \cdots; Z_{(j-1)0}; Z'_{jm_j}; Z_{(j+1)0}; \cdots; Z_{s0}; f\right] \prod_{k=m}^{m_j} (z_j - z_{jk}) . \end{aligned} \tag{39}$$

If we multiply each line in (39) by $\prod_{k=0}^{m-1}(z_j - z_{jk})$ and sum for $j = 1, \dots, s$, then we obtain

$$\begin{aligned} &\sum_{j=1}^{s}[Z_{10}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}; f] \prod_{k=0}^{m-1} (z_j - z_{jk}) \\ &= \sum_{j=1}^{s} \sum_{i_j=m}^{m_j} \left[Z_{10}; \cdots; Z_{(j-1)0}; Z_{ji_j}; Z_{(j+1)0}; \cdots; Z_{s0}; f\right] \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \\ &+ R_{1I}(z_1, \dots, z_s). \end{aligned} \tag{40}$$

By (36) and (40) Lemma 3.2 follows. □

If we separate the term for $j = 1$, in the third line in (38), we obtain

$$\begin{aligned} &\mathrm{L}_{1I}(z_1, \dots, z_s) + R_{1I}(z_1, \dots, z_s) \\ &= \sum_{\substack{i_1+\cdots+i_s \leq m \\ 0 \leq i_j \leq m-1,\ j=1,\dots,s}} [Z_{1i_1}; \cdots; Z_{si_s}; f] \prod_{j=1}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \end{aligned} \tag{41}$$

$$+ \left[Z'_{1(m-1)}; Z_{20}; \cdots; Z_{s0}; f\right] \prod_{k=0}^{m-1} (z_1 - z_{1k}) \tag{42}$$

$$+ \sum_{j=2}^{s} \left[Z_{10}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}; f\right] \prod_{k=0}^{m-1} (z_j - z_{jk}) . \tag{43}$$

Similarly, if we separate the term for $h = 1$ in R_{2I}, we obtain

$$\begin{aligned} &R_{2I}(z_1, \dots, z_s) \\ &= \sum_{\substack{i_1+\dots+i_s=m \\ 0 \leq i_j \leq m-1 j=1,\dots,s}} \left[Z'_{1i_1}; Z_{2i_2}; \cdots; Z_{si_s}; f\right] \prod_{l=0}^{i_1} (z_1 - z_{1l}) \prod_{j=2}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \end{aligned} \tag{44}$$

$$+ \sum_{j=2}^{s} \left[Z'_{10}; Z_{20}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}\right] (z_1 - z_{10}) \prod_{k=0}^{m-1} (z_j - z_{jk}) \tag{45}$$

$$+\sum_{h=2}^{s}\left(\sum_{\substack{i_h+\cdots+i_s=m\\ 0\le i_j\le m-1,\ j=h,\ldots,s}}\left[z_1;\cdots;z_{h-1};Z'_{hi_h};Z_{(h+1)i_{h+1}};\cdots;Z_{si_s};f\right]\right.$$

$$\times\prod_{l=0}^{i_h}(z_h-z_{hl})\prod_{h+1}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk}) \tag{46}$$

$$+\sum_{j=h+1}^{s}\left[z_1;\cdots;z_{h-1};Z'_{h0};Z_{(h+1)0};\cdots;Z_{(j-1)0};Z'_{j(m-1)};Z_{(j+1)0};\cdots;Z_{s0}\right]$$

$$\left.\times(z_h-z_{h0})\prod_{k=0}^{m-1}(z_j-z_{jk})\right).$$

Now, we evaluate the sum of the terms in (41), (42) and (44). We write up (41) in the form:

$$\sum_{\substack{0<i_2+\cdots+i_s\le m\\ 0\le i_j\le m-1,\ j=2,\ldots,s}}\left(\sum_{i_1=0}^{m-i_2-\cdots-i_s}[Z_{1i_1};Z_{2i_2};\cdots;Z_{si_s};f]\prod_{l=0}^{i_1-1}(z_1-z_{1l})\right)\prod_{j=2}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk}) \tag{47}$$

$$+\sum_{i_1=0}^{m-1}[Z_{1i_1};Z_{20};\cdots;Z_{s0};f]\prod_{l=0}^{i_1-1}(z_1-z_{1l}). \tag{48}$$

We also write up (44) in the following form:

$$\sum_{\substack{0<i_2+\ldots+i_s\le m\\ 0\le i_j\le m-1 j=2,\ldots,s}}\left(\left[Z'_{1(m-i_1-\cdots-i_s)};Z_{2i_2};\cdots;Z_{si_s};f\right]\prod_{l=0}^{i_1}(z_1-z_{1l})\right)\prod_{j=2}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk}). \tag{49}$$

So we have to add the formulas in (47), (48), (49) and (42). The sum of (48) and (42) equals (by the application of formula (26)):

$$[z_1;Z_{20};\cdots;Z_{s0};f]. \tag{50}$$

On the other hand, the sum of (47) and (49) equals (add first the terms in the parentheses):

$$\sum_{\substack{0<i_2+\cdots+i_s\le m\\ 0\le i_j\le m-1,\ j=2,\ldots,s}}[z_1;Z_{2i_2};\cdots;Z_{si_s};f]\prod_{j=2}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk}). \tag{51}$$

The sum of (50) and (51), the result of this step in the proof, is equal to

$$\sum_{\substack{i_2+\cdots+i_s\le m\\ 0\le i_j\le m-1,\ j=2,\ldots,s}}[z_1;Z_{2i_2};\cdots;Z_{si_s};f]\prod_{j=2}^{s}\prod_{k=0}^{i_j-1}(z_j-z_{jk}). \tag{52}$$

The next step is the evaluation of the sum of (43) and (45). If we consider the jth terms in (43) and (45), then we see that, without the factor $\prod_{k=0}^{m-1}(z_j - z_{jk})$, the sum of the two terms equals (again, by formula (26)):

$$[z_1; Z_{20}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}; f]. \tag{53}$$

Thus, the sum of (43) and (45) is

$$\sum_{j=2}^{s}[z_1; Z_{20}; \cdots; Z_{(j-1)0}; Z'_{j(m-1)}; Z_{(j+1)0}; \cdots; Z_{s0}; f] \prod_{k=0}^{m-1}(z_j - z_{jk}). \tag{54}$$

The result so far is that $L_I(z_1, \ldots, z_s) + R_{1I}(z_1, \ldots, z_s) + R_{2I}(z_1, \ldots, z_s)$ is equal to the sum of (52), (54) and (46). The sum of (52) and (54) is equal to $\widetilde{L}_J + \widetilde{R}_{1J}$, while (46) is equal to $\widetilde{R}_{2J}$, where J is similarly defined as I in connection with $i_2, \ldots, i_s$ and the function is the $s-1$-variate function $f(z_1, z_2, \ldots, z_s)$, where $z_1 \in Z$, fixed. If we assume that (35) is true for any $s-1$-variate function, then, by the above reasoning, (35) follows for the s-variate function f. □

4 Bounds When Moments of Total Order Up to m and Some Higher Order Univariate Moments are Known

In this section we assume that, in addition to all moments $\mu_{\alpha_1 \ldots \alpha_s}$, $\alpha_1 + \cdots + \alpha_s \le m$, we know the moments $E(X_j^{\beta_j})$, $\beta_j = 1, \ldots, m_j$ where $m \le m_j \le n_j$, $j = 1, \ldots, s$. If we use our notation for the multivariate moments, then we can write

$$E(X_j^{\beta_j}) = \mu_{0 \ldots 0 \beta_j 0 \ldots 0}, \; j = 1, \ldots, s,$$

where on the right hand side β_j is the jth subscript. Let H be the set given in the parentheses of (16).

As regards the ordering of the elements in the sets $Z_1, \ldots, Z_s$ we mention separately in each theorem of this section what is our assumption about it.

We keep the assumption that $K_j \subset \{m, m+1, \ldots, n_j\}$ and introduce four different structures for them as follows:

$$\begin{array}{lcc} & |K_j| \; even & |K_j| \; odd \\ min & u^{(j)}, u^{(j)}+1, \ldots, v^{(j)}, v^{(j)}+1 & m, u^{(j)}, u^{(j)}+1, \ldots, v^{(j)}, v^{(j)}+1 \\ max & m, u^{(j)}, u^{(j)}+1, \ldots, v^{(j)}, v^{(j)}+1, n_j & u^{(j)}, u^{(j)}+1, \ldots, v^{(j)}, v^{(j)}+1, n_j. \end{array} \tag{55}$$

We prove the following

Theorem 4.1 *Let $z_{j0} < z_{j1} < \cdots < z_{jn_j}$, $j = 1, \ldots, s$. Suppose that the function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ has nonnegative divided differences of total order $m+1$, and, in addition, in each variable z_j it has nonnegative divided differences of order $m + |K_j|$, where the set K_j has one of the min structures in (55).*

Under these conditions $L_I(z_1,\ldots,z_s)$, defined by (31), is a unique H-type Lagrange polynomial on Z_I and satisfies the relations

$$f(z_1,\ldots,z_s) \geq L_I(z_1,\ldots,z_s),\ (z_1,\ldots,z_s) \in Z, \tag{56}$$

i.e., the set of columns $\widehat{B}$ of $\widehat{A}$ in problem (7), with the subscript set I, is a dual feasible basis in the minimization problem (7), and

$$E[f(X_1,\ldots,X_s)] \geq E[L_I(X_1,\ldots,X_s)]. \tag{57}$$

If $\widehat{B}$ is also a primal feasible basis in problem (7), then the inequality (57) is sharp.

If all the above mentioned divided differences are nonpositive, then (56) and (57) hold with reversed inequality signs.

Proof. The unicity of the H-type Lagrange polynomial (36), and the fact that $\widehat{B}$ is a basis in the LP (7), can be proved as follows. The columns in problem (7) that correspond to the points in Z_I, form a square matrix. The fact that ($\boldsymbol{z} = (z_1,\ldots,z_s)$) $L_I(\boldsymbol{z}) = f(\boldsymbol{z})$ for $\boldsymbol{z} \in Z_I$, tells us that $\boldsymbol{f}_{\widehat{B}}$ can be represented as suitable linear combination of the rows of $\widehat{B}$. Since it holds for any function f, hence for any $\boldsymbol{f}_{\widehat{B}}$, it follows that $\widehat{B}$ must be nonsingular. This implies the unicity of the Lagrange polynomial as well.

The equivalence of the dual feasibility of $\widehat{B}$ in the minimization problem (7) and relations (56) can be deduced similarly as we did it at the end of Section 1 for problems (7) and (6).

To prove (56) we look at equation (35). Since $R_I(z_1,\ldots,z_s) = R_{1I}(z_1,\ldots,z_s) + R_{2I}(z_1,\ldots,z_s)$, it is enough to prove that $R_{1I}(z_1,\ldots,z_s) \geq 0$, $R_{2I}(z_1,\ldots,z_s) \geq 0$ for $(z_1,\ldots,z_s) \in Z$.

As regards $R_{1I}(z_1,\ldots,z_s)$, given by (33), the special structure of K_j implies that

$$\prod_{k\in\{0,\ldots,m-1\}\cup K_j} (z_j - z_{jk}) > 0 \text{ for } j \notin \{0,\ldots,m-1\} \cup K_j \tag{58}$$

and if $j \in \{0,\ldots,m-1\} \cup K_j$, the above product is 0. Since the function f has nonnegative divided differences of order $m + |K_j|$ in the variable z_i, $j = 1,\ldots,s$, it follows that for any $(z_1,\ldots,z_s) \in Z$ we have $R_{1I}(z_1,\ldots,z_s) \geq 0$.

As regards $R_{2I}(z_1,\ldots,z_s)$, defined by (34), all divided differences in the sums are of total order $m+1$ and the products that multiply them are all nonnegative for any $(z_1,\ldots,z_s) \in Z$. Thus, $R_{2I}(z_1,\ldots,z_s) \geq 0$ for any $(z_1,\ldots,z_s) \in Z$. This proves (56).

Inequality (57) is a straightforward consequence of the inequalities (56). Finally, if $\widehat{B}$ is both primal and dual feasible basis in problem (7), it is an optimal basis and the optimum value equals

$$\begin{aligned}
&min E[f(z_1,\ldots,z_s)]\\
&= f_{\widehat{B}}^T \boldsymbol{p}_{\widehat{B}} = f_{\widehat{B}}^T \widehat{B}^{-1}\widehat{\boldsymbol{b}}\\
&= f_{\widehat{B}}^T \widehat{B}^{-1} E[\widehat{\boldsymbol{b}}(X_1,\ldots,X_s)]\\
&= E[f_{\widehat{B}}^T \widehat{B}^{-1}\widehat{\boldsymbol{b}}(X_1,\ldots,X_s)]\\
&= E[L_I(X_1,\ldots,X_s)].
\end{aligned}$$

Thus, the theorem is proved. □

In the next theorem we prove both lower and upper bounds for the function $f(z_1,\dots,z_s)$, $(z_1,\dots,z_s)\in Z$ and the expectation $E[f(X_1,\dots,X_s)]$.

Theorem 4.2 *Let $z_{j0} > z_{j1} > \cdots > z_{jn_j}$, $j=1,\dots,s$. Suppose that the function $f(\boldsymbol{z})$, $\boldsymbol{z}\in Z$ has nonnegative divided differences of total order $m+1$, and, in addition, in each variable z_j it has nonnegative divided differences of order $m+|K_j|$, where K_j has one of the structures in (55) that we specify below. Under these conditions we have the following assertions:*

(a) If $m+1$ is even, $|K_j|$ is even and K_j has the max structure in (55) or $m+1$ is even, $|K_j|$ is odd and K_j has the min structure in (55), then the Lagrange polynomial $L_I(z_1,\dots,z_s)$, defined by (31), satisfies

$$f(z_1,\dots,z_s)\ge L_I(z_1,\dots,z_s),\ (z_1,\dots,z_s)\in Z, \tag{59}$$

i.e., the set of columns $\widehat{B}$ in $\widehat{A}$, corresponding to the subscripts I, is a dual feasible basis in the minimization problem (7). We also have the inequality

$$E[f(X_1,\dots,X_s)]\ge E[L_I(X_1,\dots,X_s)]. \tag{60}$$

If $\widehat{B}$ is also a primal feasible basis in the LP (7), then the lower bound (60) for $E[f(X_1,\dots,X_s)]$ is sharp.

(b) If $m+1$ is odd, $|K_j|$ is even and K_j has the max structure in (55) or $m+1$ is odd, $|K_j|$ is odd and K_j has the min structure in (55), then the Lagrange polynomial, defined by (31), satisfies

$$f(z_1,\dots,z_s)\le L_I(z_1,\dots,z_s),\ (z_1,\dots,z_s)\in Z, \tag{61}$$

i.e., the basis $\widehat{B}$ is dual feasible in the maximization problem (7). We also have the inequality

$$E[f(X_1,\dots,X_s)]\le E[L_I(X_1,\dots,X_s)]. \tag{62}$$

If $\widehat{B}$ is also a primal feasible basis in the LP (7), then the upper bound (62) for $E[f(X_1,\dots,X_s)]$ is sharp.

Proof. We prove the first part of (a), the other proofs can be carried out in the same way.

We have already shown in the proof of Theorem 4.1 that $\widehat{B}$ is a basis in the LP (7). Also, we have clarified that (59) is equivalent to the dual feasibility of $\widehat{B}$ in the minimization problem (7).

We only have to prove (59), because (60) is a trivial consequence of it and the proof of the sharpness of (60), i.e., the primal feasibility of $\widetilde{B}$, is the same as that in the proof of Theorem 4.1.

We prove that $R_{1I} \geq 0$ and $R_{2I} \geq 0$ for all $(z_1, \ldots, z_s) \in Z$. The nonnegativity of R_{1I} follows from the fact that each term in the sum of R_{1I} is the product of a nonnegative divided difference and some

$$\prod_{k \in \{0,\ldots,m-1\} \cup K_j} (z_j - z_{jk}). \tag{63}$$

Since $m+1$ is even, we have the inequality

$$\prod_{k \in \{0,\ldots,m-1\}} (z_j - z_{jk}) \leq 0. \tag{64}$$

The product in (64) is zero, if $0 \leq j \leq m-1$. On the other hand, due to the special structure of K_j, we also have for $j \geq m$:

$$\prod_{k \in K_j} (z_j - z_{jk}) \leq 0. \tag{65}$$

Thus, $R_{1I}(z_1, \ldots, z_s) \geq 0$ for any $(z_1, \ldots, z_s) \in Z$.

The nonnegativity of R_{2I} follows from the fact that each term in the sum that defines it is the product of a nonnegative divided difference and an even number of factors of the form $z_j - z_{jk} \leq 0$. Thus, $R_{2I}(z_1, \ldots, z_s) \geq 0$ for any $(z_1, \ldots, z_s) \in Z$ and the theorem is proved. □

In the next theorem we use the subscript set

$$\begin{aligned} I = &\ I_0 \cup \left(\cup_{j=1}^{s} I_j \right), \text{ where} \\ I_0 = &\ \{(i_1, \ldots, i_s) |\ i_j \text{ integer},\ 0 \leq n_j - i_j \leq m-1,\ j = 1, \ldots, s, \\ &\ n_1 - i_1 + \cdots + n_s - i_s \leq m\}, \\ I_j = &\ \{(i_1, \ldots, i_s) |\ (n_j - i_j) \in K_j,\ i_l = 0\ l \neq j\},\ j = 1, \ldots, s. \end{aligned} \tag{66}$$

The Lagrange polynomial corresponding to Z_I is:

$$\begin{aligned} & L_I(z_1, \ldots, z_s) = \\ & \sum_{\substack{i_1 + \ldots + i_s \leq m \\ 0 \leq i_j \leq m-1,\ j=1,\ldots,s}} \left[z_{1n_1}, \ldots, z_{1(n_1-i_1)}; \cdots; z_{sn_s}, \ldots, z_{s(n_s-i_s)}; f \right] \prod_{j=1}^{s} \prod_{k=n_j-i_j+1}^{n_j} (z_j - z_{jk}) + \\ & + \sum_{j=1}^{s} \sum_{i_j=1}^{|K_j|} \Big[z_{1n_1}; \cdots; z_{(j-1)n_{j-1}}; z_{jn_j}, \ldots, z_{j(m-1)}, z_{j(n_j - k_j^{(1)})}, \ldots, z_{j(n_j - k_j^{(i_j)})}; \\ & \qquad z_{(j+1)n_{j+1}}; \cdots; z_{sn_s}; f \Big] \times \\ & \qquad \times \prod_{k=0}^{n_j-m+1} (z_j - z_{jk}) \prod_{l=1}^{i_j-1} (z_j - z_{j(n_j-k_j^{(l)})}). \end{aligned} \tag{67}$$

Theorem 4.3 *Let $z_{j0} < z_{j1} < \cdots < z_{jn_j}$, $j = 1, \ldots, s$. Suppose that the function $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ has nonnegative divided differences of total order $m+1$, and, in addition, in each variable z_j it has nonnegative divided differences of order $m + |K_j|$, where $n_j - K_j$ has one of the structures in (55) that we specify below. Under these conditions we have the following assertions:*

(a) *If $m+1$ is even, $|K_j|$ is even and $n_j - K_j$ has the max structure in (55), or $m+1$ is even $|K_j|$ is odd and $n_j - K_j$ has the min structure in (55), then the Lagrange polynomial $L_I(z_1, \ldots, z_s)$, defined by (67), satisfies*

$$f(z_1, \ldots, z_s) \geq L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{68}$$

i.e., the set of those columns of $\widehat{A}$ in problem (7) that correspond to the subscripts in I, is dual feasible in the minimization problem (7). We also have the inequality

$$E[f(X_1, \ldots, X_s)] \geq E[L_I(X_1, \ldots, X_s)]. \tag{69}$$

If $\widehat{B}$ is also a primal feasible basis in problem (7), then the bound in (69) is sharp.

(b) *If $m+1$ is odd, $|K_j|$ is even and $n_j - K_j$ has a max structure in (55), or $m+1$ is odd, $|K_j|$ is odd and $n_j - K_j$ has a min structure in (55), then $L_I(z_1, \ldots, z_s)$, satisfies*

$$f(z_1, \ldots, z_s) \leq L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{70}$$

i.e., $\widehat{B}$ is a dual feasible basis in the maximization problem (7). We also have the inequality

$$E[f(X_1, \ldots, X_s)] \leq E[L_I(X_1, \ldots, X_s)]. \tag{71}$$

If $\widehat{B}$ is also primal feasible basis in problem (7), then the bound in (71) is sharp.

Proof. The assertion that $\widehat{B}$ is a basis can be proved in the usual way. Otherwise, the theorem is a consequence of Theorem 4.2, if we replace $z_{j(n_j-0)}, z_{j(n_j-1)}, \ldots, z_{j(n_j-n_j)}$ for $z_{j0}, z_{j1}, \ldots, z_{jn_j}$, $i = 1, \ldots, s$ and $(z_1, \ldots, z_s) \in Z$. □

The next theorem presents bounds for $E[f(X_1, \ldots, X_s)]$ in the case where in connection with each variable X_j, $j = 1, \ldots, s$ we know the expectation, variance, skewness and kurtosis, or we know the first four moments $E(X_j)$, $E(X_j^2)$, $E(X_j^3)$, $E(X_j^4)$, further, in addition, we know all covariances $Cov(X_i, X_j)$, $i \neq j$.

Theorem 4.4 *Let $z_{j0} < z_{j1} < \cdots < z_{jn_j}$ $j = 1, \ldots, s$. Suppose that the function $f(\boldsymbol{z})$ $\boldsymbol{z} \in Z$ has nonnegative divided differences of total order $m+1 = 3$, and, in addition, in each variable z_j it has nonnegative divided differences of order $m+3 = 5$. Then we have the following assertions.*

(a) *If $|K_j| = 3$ and each K_j consists of m and any two consecutive elements of $\{m+1, m+2, \ldots, n_j\}$, $j = 1, \ldots, s$ (i.e., K_j has the min structure in (55)) and I is the subscript set (28)-(30), then the Lagrange polynomial (31) satisfies*

$$f(z_1, \ldots, z_s) \geq L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{72}$$

i.e., the set of columns $\widehat{B}$ of $\widehat{A}$ in problem (7), that correspond to the subscript set I in (28), is a dual feasible basis in problem (7). We also have the inequality

$$E[f(X_1, \ldots, X_s)] \geq E[L_I(X_1, \ldots, X_s)]. \tag{73}$$

If $\widehat{B}$ is also a primal feasible basis in problem (7), then the inequality in (73) is sharp.

(b) If $|K_j| = 3$ and each $n_j - K_j$ consists of m and any two consecutive elements of $\{m+1,\ldots,n_j\}$, $j = 1,\ldots,s$ (i.e., $n_j - K_j$ has the min structure in (55)) and I is the subscript set (66), then the Lagrange polynomial (31) satisfies

$$f(z_1,\ldots,z_s) \le L_I(z_1,\ldots,z_s),\ (z_1,\ldots,z_s) \in Z, \tag{74}$$

i.e., the set of columns $\widehat{B}$ of $\widehat{A}$ in problem (7), that corresponds to the subscript set I in (28), is a dual feasible basis in problem (7). We also have the inequality

$$E[f(X_1,\ldots,X_s)] \le E[L_I(X_1,\ldots,X_s)]. \tag{75}$$

If $\widehat{B}$ is also a primal feasible basis in problem (7), then the inequality in (75) is sharp.

Proof. The theorem is an immediate consequence of Theorems 4.1 and 4.3. □

Remark 4.1 *For the case of $s = 2$ the Lagrange polynomial in Theorem 4.4, Case (a) has the detailed form:*

$$\begin{aligned}
&L_I(z_1,z_2)\\
&= [z_{10}; z_{20}; f] + [z_{10}, z_{11}; z_{20}; f](z_1 - z_{10})\\
&+[z_{10}; z_{20}, z_{21}; f](z_2 - z_{20})\\
&+[z_{10}, z_{11}; z_{20}, z_{21}; f](z_1 - z_{10})(z_2 - z_{20})\\
&+[z_{10}, z_{11}, z_{12}; z_{20}; f](z_1 - z_{10})(z_1 - z_{11})\\
&+[z_{10}, z_{11}, z_{12}, z_{1i}; z_{20}; f](z_1 - z_{10})(z_1 - z_{11})(z_1 - z_{12})\\
&+[z_{10}, z_{11}, z_{12}, z_{1i}, z_{1(i+1)}; z_{20}; f](z_1 - z_{10})(z_1 - z_{11})(z_1 - z_{12})(z_1 - z_{1i})\\
&+[z_{10}; z_{20}, z_{21}, z_{22}; f](z_2 - z_{20})(z_2 - z_{21})\\
&+[z_{10}; z_{20}, z_{21}, z_{22}, z_{2k}; f](z_2 - z_{20})(z_2 - z_{21})(z_2 - z_{22})\\
&+[z_{10}; z_{20}, z_{21}, z_{22}, z_{2k}, z_{2(k+1)}; f](z_2 - z_{20})(z_2 - z_{21})(z_2 - z_{22})(z_1 - z_{2k}).
\end{aligned} \tag{76}$$

For the case of $s = 2$ the Lagrange polynomial in Theorem 4.4, Case (b) has the detailed form:

$$\begin{aligned}
&L_I(z_1,z_2)\\
&= [z_{1n_1}; z_{2n_2}; f] + [z_{1n_1}, z_{1(n_1-1)}; z_{2n_2}; f](z_1 - z_{1n_1})\\
&+[z_{1n_1}; z_{2n_2}, z_{2(n_2-1)}; f](z_2 - z_{2n_2})\\
&+[z_{1n_1}, z_{1(n_1-1)}; z_{20}, z_{2(n2-1)}; f](z_1 - z_{1n_1})(z_2 - z_{2n_2})\\
&+[z_{1n_1}, z_{1(n_1-1)}, z_{1(n_1-2)}; z_{2n_2}; f](z_1 - z_{1n_1})(z_1 - z_{1(n_1-1)})\\
&+[z_{1n_1}, z_{1(n_1-1)}, z_{1(n1-2)}, z_{1i}; z_{2n_2}; f](z_1 - z_{1n_1})(z_1 - z_{1(n_1-1)})(z_1 - z_{1(n_1-2)})\\
&+[z_{1n_1}, z_{1(n_1-1)}, z_{1(n_1-2)}, z_{1i}, z_{1(i+1)}; z_{2n_2}; f](z_1 - z_{1n_1})(z_1 - z_{1(n_1-1)})(z_1 - z_{1(n_1-2)})(z_1 - z_{1i})\\
&+[z_{1n_1}; z_{2n_2}, z_{2(n_2-1)}, z_{2(n_2-2)}; f](z_2 - z_{2n_2})(z_2 - z_{2(n_2-1)})\\
&+[z_{1n_1}; z_{2n_2}, z_{2(n_2-1)}, z_{2(n2-2)}, z_{2i}; f](z_2 - z_{2n_2})(z_2 - z_{2(n_2-1)})(z_2 - z_{2(n_2-2)})\\
&+[z_{1n_1}; z_{2n_2}, z_{2(n_2-1)}, z_{2(n_2-2)}, z_{2k}, z_{2(k+1)}; f](z_2 - z_{2n_2})(z_2 - z_{2(n_2-1)})(z_2 - z_{2(n_2-2)})(z_2 - z_{2k}).
\end{aligned} \tag{77}$$

If we replace X_1, X_2 for z_1, z_2, respectively, in (76) and (77), and take expectations, then the value resulting from (76) (from (77)) provides us with a lower (upper) bound for

$E[f(X_1, X_2)]$. *Note that all expectations in (76) and (77) can be expressed by the use of the moments*

$$E(X_j^k),\ \ k = 1, 2, 3, 4,\ \ j = 1, 2$$

and the covariance

$$Cov(X_1, X_2).$$

Given a dual feasible basis, we may look at it as an initial basis and carry out the dual algorithm of linear programming to obtain the best possible bound. The knowledge of an initial dual feasible basis has two main advantages. First it saves roughly half of the running time of the entire dual algorithm. Second, it improves on the numerical accuracy of the computation that we carry out in connection with our LP's.

5 More Dual Feasible Bases, Algorithms and Bounds in the Bivariate Case

In the bivariate case we can create a larger variety of dual feasible bases for problem (7), and produce better bounds than what we can obtain by the use of the dual feasible basis structures presented in the previous section. We drop the condition that the elements of the supports of the random variables X_1, X_2 are arranged in increasing order, we only assume that each set $Z_1 = \{z_{10}, \ldots, z_{1n_1}\}$, $Z_2 = \{z_{20}, \ldots, z_{2n_2}\}$ consist of distinct elements.

For convenience we write up the Lagrange polynomial (31) and the residual terms (33), (34) for the case of $s = 2$. We obtain:

$$\begin{aligned} & L_I(z_1, z_2) \\ &= \sum_{\substack{i_1 + i_2 \le m \\ 0 \le i_j \le m-1,\ j=1,2}} [z_{10}, \ldots z_{1i_1}; z_{20}, \ldots, z_{2i_2}; f] \prod_{j=1}^{2} \prod_{k=0}^{i_j - 1} (z_j - z_{jk}) \\ &+ \sum_{i=1}^{|K_1|} [z_{10}, \ldots, z_{1(m-1)}, z_{1k_1^{(1)}}, \ldots, z_{1k_1^{(i)}}; z_{20}; f] \prod_{k \in \{0, \ldots, m-1, k_1^{(1)}, \ldots, k_1^{(i)}\}} (z_1 - z_{1k}) \\ &+ \sum_{i=1}^{|K_2|} [z_{10}; z_{20}, \ldots, z_{2(m-1)}, z_{2k_2^{(1)}}, \ldots, z_{2k_2^{(i)}}; f] \prod_{k \in \{0, \ldots, m-1, k_2^{(1)}, \ldots, k_2^{(i)}\}} (z_2 - z_{2k}), \end{aligned} \tag{78}$$

$$\begin{aligned} & R_{1I}(z_1, z_2) \\ &= [z_{10}, \ldots, z_{1(m-1)}, Z_{1K_1}, z_1; z_{20}; f] \prod_{k \in \{0, \ldots, m-1\} \cup K_1} (z_1 - z_{1k}) \\ &+ [z_{10}; z_{20}, \ldots, z_{2(m-1)}, Z_{2K_2}, z_2; f] \prod_{k \in \{0, \ldots, m-1\} \cup K_2} (z_2 - z_{2k}), \end{aligned} \tag{79}$$

$$
\begin{aligned}
&R_{2I}(z_1,z_2)\\
=&\sum_{\substack{i_1+i_2=m\\ 0\le i_j\le m_1,\ j=1,2}}[z_{10},\ldots,z_{1i_1},z_1;z_{20},\ldots,z_{2i_2};f]\prod_{l=0}^{i_1}(z_1-z_{1l})\prod_{k=0}^{i_2-1}(z_2-z_{2k})\\
&+[z_{10},z_1;z_{20},\ldots,z_{2(m-1)},z_2;f](z_1-z_{10})\prod_{k=0}^{m-1}(z_2-z_{2k}).
\end{aligned}
\tag{80}
$$

We want to ensure that the Lagrange polynomial corresponding to the set Z_I, i.e., the polynomial (78) should satisfy

$$L_I(z_1,z_2)\le f(z_1,z_2),\ (z_1,z_2)\in Z \tag{81}$$

or

$$L_I(z_1,z_2)\ge f(z_1,z_2),\ (z_1,z_2)\in Z. \tag{82}$$

A sufficient condition for (81) ((82)) is that $R_{1I}(z_1,z_2)\ge 0$, $R_{2I}\ge 0$, for all $(z_1,z_2)\in Z$ ($R_{1I}(z_1,z_2)\le 0$, $R_{2I}\le 0$, for all $(z_1,z_2)\in Z$).

All coefficients in the expression of $R_{1I}(z_1,z_2)$ and $R_{2I}(z_1,z_2)$ are divided differences of order $m+1$. Assume all of them are nonnegative. Hence, in order to ensure (81) ((82)) we have to choose I in such a way that all products in (79) and (80) be nonnegative (nonpositive).

Consider the $m\times(m+1)$ array

$$
\begin{array}{ccccccc}
z_{10} & z_{11} & z_{12} & \cdots & z_{1(m-2)} & z_{1(m-1)} & z_{20}\\
z_{10} & z_{11} & z_{12} & \cdots & z_{1(m-2)} & z_{20} & z_{21}\\
 & & & \vdots & & & \\
z_{10} & z_{11} & z_{20} & \cdots & z_{2(m-4)} & z_{2(m-3)} & z_{2(m-2)}\\
z_{10} & z_{20} & z_{21} & \cdots & z_{2(m-3)} & z_{2(m-2)} & z_{2(m-1)}
\end{array}
\tag{83}
$$

and associate each of the first $m-1$ rows with the corresponding product in the second line of (80). Similarly, associate the last row of (83) with the product in the third line of (80) that defines $R_{2I}(z_1,z_2)$. A sufficient condition for the nonnegativity of all products in (80), for all $(z_1,z_2)\in Z$, is that

$$
\begin{aligned}
&|\{i|0\le i\le i_1,\ z_{1i}>z_1\}|\\
+&|\{i|0\le i\le i_2,\ z_{2i}>z_2\}| \ = \text{even number}
\end{aligned}
\tag{84}
$$

should hold for all $(z_1,z_2)\in Z$ in each row of (83), i.e., for every $i_1\ge 0,\ i_2\ge 0$ integers satisfying $i_1+i_2=m-1$. Similarly, a sufficient condition for the nonpositivity of all products in (80), for all $(z_1,z_2)\in Z$, is that

$$
\begin{aligned}
&|\{i|0\le i\le i_1,\ z_{1i}>z_1\}|\\
+&|\{i|0\le i\le i_2,\ z_{2i}>z_2\}| \ = \text{odd number}
\end{aligned}
\tag{85}
$$

should hold for all $(z_1,z_2)\in Z$ in each row of (83), i.e., for every $i_1\ge 0,\ i_2\ge 0$ integers satisfying $i_1+i_2=m-1$.

Consider first the case, where we want to construct lower bound, i.e., satisfy relations (81), by suitable choices of $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$. We present an algorithm to find these sequences. We may assume, without loss of generality, that the ordered sets Z_1 and Z_2 are the following: $Z_1 = \{0, 1, \ldots, n_1\}$, $Z_2 = \{0, 1, \ldots, n_2\}$.

Min Algorithm

Algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$ satisfying (84).

Step 0. Initialize $t = 0$, $-1 \le q \le m-1$, $L = \{0, 1, \ldots, q\}$, $U = \{n_1, n_1 - 1, \ldots, n_1 - (m-q-2)\}$, $V^0 = \{$arbitrary merger of the sets $L, U\} = \{v^0, v^1, \ldots, v^{m-1}\}$. If $|U|$ is even, then $h^0 = 0$, $l^0 = 1$, $u^0 = n_2$, and if $|U|$ is odd, then $h^0 = n_1$, $l^0 = 0$, $u^0 = n_2 - 1$. Go to Step 1.

Comment: The first m elements of the first row in (83) are the elements of V^0, the $m+1st$ element of the same row is h^0. All the sets L, U, V^0 are ordered.

Step 1. If $t = m$, then go to Step 3. Otherwise go to Step 2.

Step 2. Let $V^t = \{v^0, v^1, \ldots, v^{m-1-t}\}$, $H^t = \{h^0, h^1, \ldots, h^t\}$ If $v^{m-1-t} \in L$, then let $h^{t+1} = l^t$, $l^{t+1} = l^t + 1$, $u^{t+1} = u^t$, and if $v^{m-1-t} \in U$, then let $h^{t+1} = u^t$, $u^{t+1} = u^t - 1$, $l^{t+1} = l^t$. Set $t \leftarrow t+1$ and go to Step 1.

Comment: The elements of V^t, H^t, in that order, constitute the tth row of tableau (83).

Step 3. Stop, all m rows of the tableau have been created. The tableau (83) has rows $\{V^t, H^t\}$, $t = 0, 1, \ldots, m-1$.

The points presented below represent those columns in problem (7) which correspond to the subscript set I_0:

$$\begin{array}{lllll}
(z_{10}, z_{20}), & (z_{11}, z_{20}), & \cdots & (z_{1(m-2)}, z_{20}), & (z_{1(m-1)}, z_{20}), \\
(z_{10}, z_{21}), & (z_{11}, z_{21}), & \cdots & (z_{1(m-2)}, z_{21}), & \\
\vdots & \vdots & & & \\
(z_{10}, z_{2(m-2)}), & (z_{11}, z_{2(m-2)}), & & & \\
(z_{10}, z_{2(m-1)}). & & & &
\end{array} \tag{86}$$

It remains to find suitable sets K_1 and K_2 to make $R_{1I}(z_1, z_2) \ge 0$, for all $(z_1, z_2) \in Z$.

Let $0, 1, \ldots, q_2, n_2, \ldots, n_2 - (m - q_2 - 2)$ be the numbers used to construct $z_{20}, z_{21}, \ldots, z_{2(m-1)}$. Then the set K_j should be taken from the set $\{q_j + 1, q_j + 2, \ldots, n_j - (m - q_j - 1)\}$, $j = 1, 2$. These subsets of the sets Z_1, Z_2, respectively, remain intact after the construction of I_0. For each j the products

$$\prod_{k=0}^{m-1} (z_j - z_{jk}), \ (z_1, z_2) \in Z \tag{87}$$

do not change sign, but they may be positive or negative, depending on the construction of $z_{j0}, \ldots, z_{j(m-1)}$, $j = 1, 2$.

If (87) is positive, then K_j should follow a minimum structure in (55), and if (87) is negative, then K_j should follow a maximum structure.

We have completed the construction of the dual feasible basis related to the subscript set I.

If we want to satisfy the relation (82), i.e., construct an upper bound, then only slight modification is needed in the above algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$. We only have to rewrite Step 0 and keep the other steps unchanged.

Max Algorithm

Step 0 of algorithm to find $z_{10}, \ldots, z_{1(m-1)}; z_{20}, \ldots, z_{2(m-1)}$ satisfying (85).

Step 0. Initialize $t = 0$, $-1 \leq q \leq m-1$, $L = \{0, 1, \ldots, q\}$, $U = \{n_1, n_1 - 1, \ldots, n_1 - (m - q - 2)\}$, $V^0 = \{\text{arbitrary merger of the sets } L, U\} = \{v^0, v^1, \ldots, v^{m-1}\}$. If $|U|$ is odd, then $h^0 = 0$, $l^0 = 1$, $u^0 = n_2$, and if $|U|$ is even, then $h^0 = n_1$, $l^0 = 0$, $u^0 = n_2 - 1$. Go to Step 1, etc.

The points, representing the basic columns in problem (7) are given by (86).

It remains to find suitable sets K_1 and K_2 to make $R_{1I}(z_1, z_2) \leq 0$ for all $(z_1, z_2) \in Z$.

The set K_j should be taken from the set $\{q_j + 1, q_j + 2, \ldots, n_j - (m - q_j - 1)\}$, $j = 1, 2$.

In case of the upper bound we have to choose K_j the other way around as in case of the Min algorithm. If (87) is positive, then K_j should follow a maximum structure, otherwise a minimum structure.

We have completed the construction of the dual feasible basis related to the subscript set I.

In the general case, where Z_1 is not necessarily $\{0, 1, \ldots, n_1\}$ and Z_2 is not necessarily $\{0, 1, \ldots, n_2\}$, we do the following. First we order the elements in both Z_1 and Z_2 in increasing order. Then, establish one-to-one correspondences between the elements of Z_1 and the elements of the set $\{0, 1, \ldots, n_1\}$ that we assume to be ordered now. We do the same to Z_2 and $\{0, 1, \ldots, n_2\}$. After that, we carry out the Min or Max Algorithm to find a dual feasible basis, using the sets $\{0, 1, \ldots, n_1\}$, $\{0, 1, \ldots, n_2\}$, as described in this section. Finally, we create the set (86), by the use of the above mentioned one-to-one correspondences.

The above construction allows for the construction of a variety of dual feasible bases. However, we do not have a simple criterion, like in the dual method, to decide which of the bases, that we can obtain by the above Min or Max Algorithm, would improve on the bound (on the value of the objective function). Still, the above construction is simple and fast, further, given the dual feasible basis and the corresponding Lagrange polynomial $L_I(z_1, z_2)$, the bound is simply $E[L_I(X_1, X_2)]$ which is not difficult to compute, at least in many cases. So, we can test a large number of dual feasible bases in a relatively short time and then choose the best one, to bound and approximate $E[f(X_1, X_2)]$. This method may produce very good results much faster than the execution of the dual algorithm.

6 Illustrative Examples

In this section we present illustrative numerical examples for discrete moment bounds. For the sake of simplicity we restrict ourselves to the bivariate case.

Example 6.1 *Let $Z_1 = Z_2 = \{0, \ldots, 9\}$, $m = 4$, $m_1 = m_2 = 6$. Based on the assumptions of Theorem 4.1, and the sets $K_1 = K_2 = \{4, 8, 9\}$, which are min structures in (55), we present a dual feasible basis for the minimum problem of (6), and a lower bound for $E[f(X_1, X_2)]$, from (57). Figure 1(a) illustrates that basis.*

We also present a dual feasible basis for the maximum problem of (6), using Theorem 4.3, and give upper bound for $E[f(X_1, X_2)]$. Let K_1, K_2 be the same as before. The basis subscript set is illustrated in Figure 1(b).

9	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
8	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
7	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
6	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
5	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
4	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
3	∗	∗	∘	∘	∘	∘	∘	∘	∘	∘
2	∗	∗	∗	∘	∘	∘	∘	∘	∘	∘
1	∗	∗	∗	∗	∘	∘	∘	∘	∘	∘
0	∗	∗	∗	∗	•	∘	∘	∘	•	•
	0	1	2	3	4	5	6	7	8	9

(a)

9	•	•	∘	∘	∘	•	∗	∗	∗	∗
8	∘	∘	∘	∘	∘	∘	∗	∗	∗	∗
7	∘	∘	∘	∘	∘	∘	∘	∗	∗	∗
6	∘	∘	∘	∘	∘	∘	∘	∘	∗	∗
5	∘	∘	∘	∘	∘	∘	∘	∘	∘	•
4	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
3	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
2	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
1	∘	∘	∘	∘	∘	∘	∘	∘	∘	•
0	∘	∘	∘	∘	∘	∘	∘	∘	∘	•
	0	1	2	3	4	5	6	7	8	9

(b)

Figure 1: $Z_1 = Z_2 = \{0, \ldots, 9\}$. $m = 4$, $m_1 = m_2 = 6$. $K_1 = K_2 = \{4, 8, 9\}$. Figure (a) illustrates a dual feasible basis for the min problem of (6), worked out from Theorem 4.1. Figure (b) illustrates a dual feasible basis for the max problem of (6), worked out from Theorem 4.3. The elements of I_0 are designated by $*$, the elements of I_1 and I_2 are designated by $\bullet$.

Consider the bivariate function

$$f(z_1, z_2) = log[(e^{\alpha z_1 + a} - 1)(e^{\beta z_2 + b} - 1) - 1], \tag{88}$$

defined for

$$e^{\alpha z_1 + a} > 2, \; e^{\beta z_2 + b} > 2,$$

where α, β are positive constants. This function is a modification of a function known as Frank's copula in actuarial mathematics (see Bowers Jr., Hickman, Jones and Nesbitt (1986)).

It is easy to see that

$$\frac{\partial f}{\partial z_j} > 0, \; \frac{\partial^2 f}{\partial z_j^2} < 0, \; \frac{\partial^3 f}{\partial z_j^3} > 0, \; \frac{\partial^4 f}{\partial z_j^4} < 0, \; \frac{\partial^5 f}{\partial z_j^5} > 0, \ldots,$$

$$j = 1, 2,$$

$$\frac{\partial^2 f}{\partial z_1 \partial z_2} < 0, \frac{\partial^3 f}{\partial z_1^2 \partial z_2} > 0, \frac{\partial^3 f}{\partial z_1 \partial z_2^2} > 0, \; etc.$$

All even (odd) total order derivatives of the function are negative (positive).

If we restrict the definition of the function to $Z = Z_1 \times Z_2$*, then it satisfies the conditions of Theorem 4.1 and 4.3. assume the following moments are known:*

μ	0	1	2	3	4	5	6
0	1	3.1855	18.5564	133.5470	1057.8635	8786.6576	74906.2014
1	3.1855	13.9179	91.0830	693.3256			
2	18.5564	91.0830	623.6111				
3	133.5470	693.3256					
4	1057.8635						
5	8786.6576						
6	74906.2014						

We can obtain lower and upper bounds for $E[f(X_1, X_2)]$*, if we calculate the value of* $\boldsymbol{f}_{\widehat{B}}^T \widehat{B}^{-1}\widehat{\boldsymbol{b}}$*, where* $\widehat{B}$ *is the matrix of the columns corresponding to the above bases. For the lower bound the result is* $-\mathbf{23.0067}$ *and for the upper bound it is* $\mathbf{8.1465}$*. There is a big gap between these bounds.*

However, the use of the algorithms of Section 5 improved on the lower bound. Below we present an example to find a dual feasible basis for the lower bound. First we run the Min Algorithm as follows.

Step 0. $t = 0$, $q = -1$, $L = \emptyset$, $U = \{9, 8, 7, 6\}$, $V^0 = U$, $|U|$ *is even, hence* h^0=0, $l^0 = 1$, $u^0 = 9$.

Step 2. $v^{m-1} \in U$, *hence* $h^1 = u^0 = 9$, $l^1 = l^0 = 0$, $u^1 = u^0 - 1 = 8$. $t = t + 1 = 1$.

Step 2. $v^{m-1} \in U$, *hence* $h^2 = u^1 = 8$, $l^2 = l^1 = 0$, $u^2 = u^1 - 1 = 7$. $t = t + 1 = 2$.

Step 2. $v^{m-1} \in U$, *hence* $h^3 = u^2 = 7$, $l^3 = l^2 = 0$, $u^3 = u^2 - 1 = 6$. $t = t + 1 = 3$.

Step 2. $v^{m-1} \in U$, *hence* $h^1 = u^0 = 9$, $l^1 = l^0 = 0$, $u^1 = u^0 - 1 = 8$. $t = t + 1 = 1$.

Step 3. $t = 3 = m - 1$, *hence we stop.*

At this point we have found $z_{10}, z_{11}, z_{12}, z_{13}$ *and* $z_{20}, z_{21}, z_{22}, z_{23}$*. We write them in the form of (83) as follows:*

9	8	7	6	0
9	8	7	0	9
9	8	0	9	8
9	0	9	8	7

To complete the dual feasible basis subscript set, we need sets $K_1 \subset \{0, \ldots, 5\}$ *and* $K_2 \subset \{1, \ldots, 6\}$*. Let* $K_1 = \{0, 1, 2\}$ *which is a suitable min structure, and* $K_2 = \{1, 2, 6\}$ *which is a suitable max structure.*

Next, we present an example to find a dual feasible basis for the upper bound. Again, first we run the Max Algorithm.

Step 0. $t = 0$, $q = -1$, $L = \emptyset$, $U = \{9, 8, 7, 6\}$, $V^0 = \{9, 8, 7, 6\}$, $|U|$ *is even, hence* $h^0 = 9$, $l^0 = 0$, $u^0 = 8$.

Step 2. $v^3 \in U$, *hence* $h^1 = u^0 = 8$, $l^1 = l^0 = 0$, $u^1 = u^0 - 1 = 7$. $t = t + 1 = 1$.

Step 2. $v^2 \in U$, *hence* $h^2 = u^1 = 7$, $l^2 = l^1 = 0$, $u^2 = u^1 - 1 = 6$. $t = t + 1 = 2$.

Step 2. $v^1 \in U$, *hence* $h^3 = u^2 = 6$, $l^3 = l^2 = 0$, $u^3 = u^2 - 1 = 5$. $t = t + 1 = 3$.
Step 1. $t = m - 1 = 3$. *Stop.*
If we write up the result in the form of (83), we obtain

$$\begin{matrix} 9 & 8 & 7 & 6 & 9 \\ 9 & 8 & 7 & 9 & 8 \\ 9 & 8 & 9 & 8 & 7 \\ 9 & 9 & 8 & 7 & 6 \end{matrix}$$

Let $K_1 = K_2 = \{0, 1, 5\}$ *which is a max structure. All dual feasible bases that can be obtained in this way have been tested. The best lower bound is* **8.0402**, *and the best upper bound is* **8.1465**.

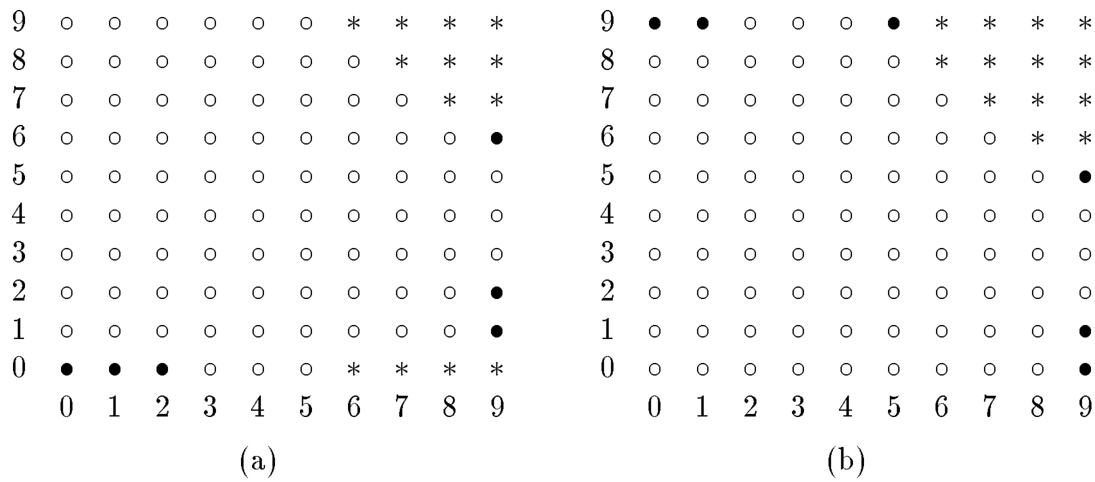

Figure 2: $Z_1 = Z_2 = \{0, \ldots, 9\}$. $m = 4$, $m_1 = m_2 = 6$. In Case (a) $K_1 = \{0, 1, 2\}$, $K_2 = \{1, 2, 6\}$, and the marked points illustrate a dual a feasible basis for the min problem of (6); in Case (b) $K_1 = K_2 = \{0, 1, 5\}$ and the marked points illustrate a dual feasible basis for the max problem of (6). The bases have been obtained by the use of the algorithms of Section 5. The elements of I_0 are designated by $*$, the elements of I_1 and I_2 are designated by $\bullet$.

Finally, we solve the problem by the dual algorithm. We can choose any of the above dual feasible bases as an initial basis, and carry out only the second stage of the method. For the above problem we have received the following results: **8.06605** *for the lower bound with basis illustrated in Figure 3(a), and* **8.1256** *for the upper bound with basis illustrated in Figure 3(b).*

Example 6.2 *Consider the function:*

$$f(z_1, z_2) = e^{\frac{z_1}{20} + \frac{z_1 z_2}{200} + \frac{z_1}{5}}, \tag{89}$$

defined on $z_1, z_2 \geq 0$. *All derivatives of this function are positive in the nonnegative orthant.*

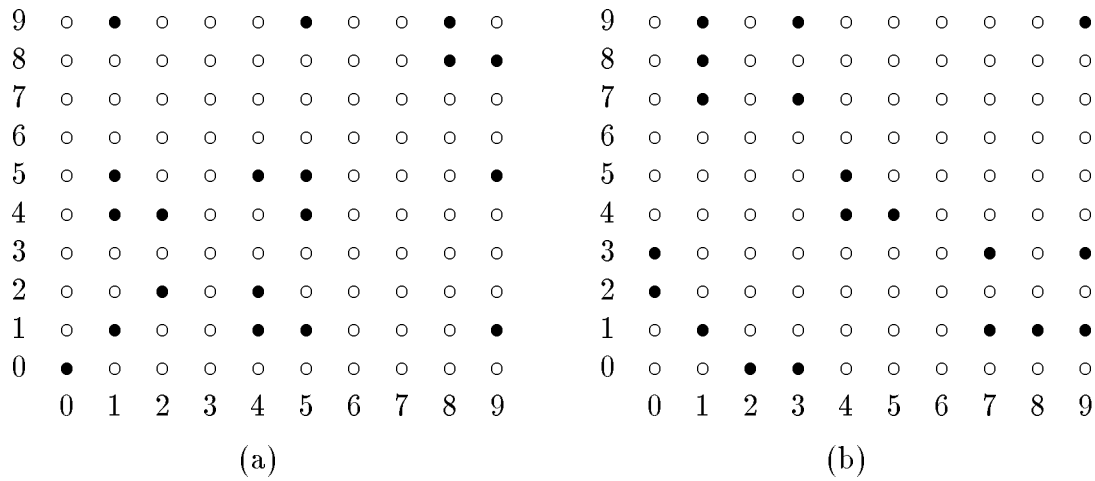

Figure 3: $Z_1 = Z_2 = \{0, \ldots, 9\}$, $m = 4$, $m_1 = m_2 = 6$. In Case (a) we have an optimal basis for the min problem of (6), in Case (b) we have one for the max problem of (6). The bases have been obtained by the dual algorithm.

Let $Z_1 = \{2, 4, \ldots, 20\}$, $Z_2 = \{0.5, 1, \ldots, 5\}$, $Z = Z_1 \times Z_2$. *Let* $m = 4$, $m_1 = m_2 = 6$, *as in example 6.1.*

It is easy to see, that the function satisfies the conditions of Theorem 4.1 and 4.3. Hence, the bases in Figure 1 are dual feasible for the min and max problems of (6), respectively. Assume the following moments are known

μ_{ij}	0	1	2	3	4	5	6
0	1	11	154	2420	40532.8	706640	12661792
1	2.75	29.5928	408.6112	6353.312			
2	9.625	102.368	1399.2232				
3	37.8125	398.8574					
4	158.33125						
5	690.078125						
6	3091.25781						

All bases of Theorems 4.1 and 4.3 have been tested for the above problem and those in Figure 1 turned out to be the best ones. The best one among these lower bounds is **3.857**, *and the best ones among these upper bounds is* **4.635**.

We can improve on both bounds by the use of the algorithms of Section 5.

First, we detail the algorithm that finds a dual feasible basis for the min problem.

Step 0. $t = 0$, $q = 1$, $L = \{0, 1\}$, $U = \{9, 8\}$, $V^0 = \{0, 9, 1, 8\}$, $|U|$ *is even, hence* $h^0 = 0$, $l^0 = 1$, $u^0 = 9$.

Step 2. $v^3 \in U$, *hence* $h^1 = u^0 = 9$, $l^1 = l^0 = 1$, $u^1 = u^0 - 1 = 8$. $t = t + 1 = 1$.

Step 2. $v^2 \in L$, *hence* $h^2 = l^1 = 1$, $l^2 = l^1 + 1 = 2$, $u^2 = u^1 = 8$. $t = t + 1 = 2$.

Step 2. $v^1 \in U$, *hence* $h^3 = u^2 = 8$, $l^3 = l^2 = 2$, $u^3 = u^2 - 1 = 7$. $t = t + 1 = 3$.

Step 1. $t = m - 1 = 3$. *Stop.*

At this point we have found the sequences V^0 and H^{m-1}. By the use of the elements of the ordered sets Z_1, Z_2, the array (83) is the following:

2	20	4	18	0.5
2	20	4	0.5	5
2	20	0.5	5	1
2	0.5	5	1	4.5

Let us choose $K_1 = K_2 = \{2, 5, 6\}$. The obtained basis is illustrated in Figure 4(a). The related bound is **3.9122**.

Now, we run the algorithm to find an upper bound.

Step 0. $t = 0$, $q = 2$, $L = \{0, 1, 2\}$, $U = \{9\}$, $V^0 = \{0, 1, 9, 2\}$, $|U|$ *is odd, hence* $h^0 = 0$, $l^0 = 1$, $u^0 = 9$.

Step 2. $v^3 \in L$, *hence* $h^1 = l^0 = 1$, $l^1 = l^0 + 1 = 2$, $u^1 = u^0 = 9$. $t = t + 1 = 1$.

Step 2. $v^2 \in U$, *hence* $h^2 = u^1 = 9$, $l^2 = l^1 = 2$, $u^2 = u^1 - 1 = 8$. $t = t + 1 = 2$.

Step 2. $v^1 \in L$, *hence* $h^3 = l^2 = 2$, $l^3 = l^2 + 1 = 3$, $u^3 = u^2 = 8$. $t = t + 1 = 3$.

Step 1. $t = m - 1 = 3$. *Stop.*

We can write up the results in the form of (83):

2	4	20	6	0.5
2	4	20	0.5	1
2	4	0.5	1	5
2	0.5	1	5	1.5

Choose $K_1 = K_2 = \{3, 6, 7\}$. The obtained basis is illustrated in Figure 4(b). The corresponding bound is **4.0103**.

The problem has been solved by the dual algorithm as well. We have obtained the following results: **3.9489** *for the lower bound and* **3.9619** *for the upper bound.*

In the next three examples, we present only the best lower and upper bounds obtained by the use of the Min and Max Algorithms of Section 5.

Example 6.3 *The problem is taken from Prékopa, Vizvári and G. Regős (1997). We have 40 events, subdivided into two 20-element groups; X_j equals the number of events that occur in the jth group, $j = 1, 2$, $Z_1 \times Z_2 = \{0, \ldots, 20\} \times \{0, \ldots, 20\}$.*

We want to find bounds for the probability that at least one out of the 40 events occurs, i.e.

$$P(X_1 + X_2 \geq 1) = E[f(X_1, X_2)],$$

where

$$f(z_1, z_2) = \begin{cases} 0, & \text{if } (z_1, z_2) = (0, 0) \\ 1, & \text{otherwise.} \end{cases} \tag{90}$$

5	∗	∗	∘	∘	∘	∘	∘	∘	∗	∗
4.5	∗	∘	∘	∘	∘	∘	∘	∘	∘	∗
4	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
3.5	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
3	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
2.5	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
2	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
1.5	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
1	∗	∗	∘	∘	∘	∘	∘	∘	∘	∗
0.5	∗	∗	•	∘	∘	•	•	∘	∗	∗
	2	4	6	8	10	12	14	16	18	20

(a)

5	∗	∗	∘	∘	∘	∘	∘	∘	∘	∗
4.5	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
4	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
3.5	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
3	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
2.5	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
2	•	∘	∘	∘	∘	∘	∘	∘	∘	∘
1.5	∗	∗	∘	∘	∘	∘	∘	∘	∘	∘
1	∗	∗	∗	∘	∘	∘	∘	∘	∘	∗
0.5	∗	∗	∗	•	∘	∘	•	•	∘	∗
	2	4	6	8	10	12	14	16	18	20

(b)

Figure 4: $Z_1 = \{2, 4, \ldots, 20\}$, $Z_2 = \{0.5, 1, \ldots, 5\}$. $m = 4$, $m_1 = m_2 = 6$.In Case (a) $K_1 = K_2 = \{2, 5, 6\}$, and the marked points illustrate a dual feasible basis for the min problem of (6). In Case (b) $K_1 = K_2 = \{3, 6, 7\}$ and the marked points illustrate a dual feasible basis for the max problem of (6), worked out from the algorithms of Section 5. The elements of I_0 are designated by $*$, the elements of I_1 and I_2 are designated by $\bullet$.

Prékopa (1998) has shown, that if $m+1$ is even (odd), then all divided differences (90) of total order $m+1$ are nonpositive (nonnegative). Suppose that we know the following cross binomial moments

1st group	*2nd group* 0	1	2	3	4	5	6
0	1.00	1.93	4.70	12.19	41.05	127.37	317.72
1	6.23	3.28	31.15	186.89	794.26	2541.64	
2	46.04	31.15	295.90	1775.41	7545.49		
3	216.09	186.89	1775.41	10652.46			
4	724.30	794.26	7545.49				
5	1848.66	2541.64					
6	3739.79.						

We have obtained the following results, by the use of the dual feasible bases of Section 5 for problem (11):

m	m_1	m_2	*lower bound*	*upper bound*
4	4	4	0.54074	1
4	6	6	0.78259	0.81423
6	6	6	0.64435	1

The best bounds correspond to the second case, where $m = 4$, $m_1 = m_2 = 6$, even though in the third case more moments are taken into account. This phenomenon is explained by the fact that in the second case we have the freedom to choose the sets K_1, K_2 arbitrarily (in agreement with (55)).

We know from Prékopa, Vizvári and Regős (1997), that the optimal value is **0.80325** *for the min, and* **0.80410** *for the max problem, in case of* $m = m_1 = m_2 = 6$. *These values have been obtained by the full execution of the dual method of linear programming.*

In the following example we present bounds in the case where in connection with each variable X_j, $j = 1, 2$ we know the expectation, variance, skewness and kurtosis, i.e., we know the first four moments and the covariance $Cov(X_1, X_2)$.

Example 6.4 *Consider the bivariate utility function (88). Let* $\alpha = \beta = 1$, $a = b = 0$ *and* $Z_1 = Z_2 = \{1, \ldots, 10\}$.

Case 1

Assume that, in addition to μ_{11}, *the following moments are known:* ($\mu_{00} = 1$), $\mu_{10} = \mu_{01} = 11/2$, $\mu_{20} = \mu_{02} = 33/2$, $\mu_{30} = \mu_{03} = 33$, $\mu_{40} = \mu_{04} = 231/5$.

The results are presented below. The lower and upper bound columns contain values obtained by the Min and Max Algorithms of Section 5. The min and max columns contain values obtained by the dual algorithm carried out for problem (7).

μ_{11}	*lower bound*	*upper bound*	*min*	*max*
30.25	10.77220	10.89995	10.7761	10.89184
35	10.77218	10.89995	10.77590	10.88837
25	10.77224	10.89995	10.8224	10.89191

Remark: In the first case $\mu_{11} = 30.25 = (11/2)^2 = \mu_{01}\mu_{10}$, *hence the two random variables do not correlate.*

Case 2

Now, suppose that ($\mu_{00} = 1$), $\mu_{10} = \mu_{01} = 2615937/625000$, $\mu_{20} = \mu_{02} = 5435467/500000$, $\mu_{30} = \mu_{03} = 108634563/5000000$, $\mu_{40} = \mu_{04} = 162205043/5000000$, $\mu_{11} = 2615937/625000$.

We have obtained the following results:

lower bound	*upper bound*	*min*	*max*
8.00326	8.1915	8.05739	8.1590

Example 6.5 *Finally, we consider the function*

$$f(z_1, z_2) = e^{\frac{z_1}{10} + \frac{z_2}{10} + \frac{z_1 z_2}{200}},$$

the support set $Z_1 \times Z_2 = \{0, \ldots, 20\} \times \{0, \ldots, 20\}$ *and the following power moments:*

μ_{ij}	0	1	2	3	4	5	6
0	1.00	6.23	98.31	1579.01	25813.23	429472.13	7269694.11
1	1.93	3.28	65.58	1311.52	26229.7	524590	
2	11.33	65.58	1311.48	26229.5	524590		
3	103.27	1311.52	26229.5	524590			
4	1491.77	26229.7	524590				
5	27107.8	524590					
6	528938						

We have obtained the following results:

m	m_1	m_2	*lower bound*	*upper bound*
3	6	6	6.000222	6.004789
4	6	6	6.003941	6.00455.

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