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Sharp Bounds for Probabilities with Given Shape Information

Ersoy Subasi Mine Subasi András Prékopa

Abstract. In [13] the authors presented a method to obtain sharp lower and upper bounds for expectations of convex functions of discrete random variables as well as probabilities that at least one out of n events occurs, based on the knowledge of some of the power moments of the random variables involved, or the binomial moments of the number of events which occur. In this paper Binomial moment problem with finite, preassigned supports and given shapes of the distribution is formulated and used to obtain sharp lower and upper bounds for probabilities that exactly r out of n events occurs as well as probabilities that at least r out of n events occurs. The bounds are based on the knowledge of some of the binomial moments of the number of events which occur. Numerical examples and applications in reliability and finance are presented.

1 Introduction

Let ξ be a random variable, the possible values of which are known to be nonnegative numbers $z_0 < z_1 < ... < z_n$. Let $p_i = P(\xi = z_i), \quad i = 0, 1, ..., n$. Suppose that these probabilities are unknown but the binomial moments $S_k = E\left[\binom{\xi}{k}\right], \ k = 1, ..., m$, where $m < n$ are known.

The starting point of our investigation is the following linear programming problem

$$\begin{aligned} &min(max)\{f(z_0)p_0 + f(z_1)p_1 + ... + f(z_n)p_n\} \\ &\text{subject to} \\ &p_0 + p_1 + ... + p_n = 1 \\ &z_0p_0 + z_1p_1 + ... + z_np_n = S_1 \\ &\binom{z_0}{2}p_0 + \binom{z_1}{2}p_1 + ... + \binom{z_n}{2}p_n = S_2 \\ &\vdots \\ &\binom{z_0}{m}p_0 + \binom{z_1}{m}p_1 + ... + \binom{z_n}{m}p_n = S_m \\ &p_0 \geq 0, p_1 \geq 0, ..., p_n \geq 0. \end{aligned} \tag{1.1}$$

Problem (1.1) is called the binomial moment problem, respectively and have been studied extensively in [1, 7, 8, 9, 10]. Let $A; a_0, a_1, ..., a_n$ and b denote the matrix of the equality constraints its columns and the vector of the right-hand side values. We will alternatively use the notation f_k instead of $f(z_k)$.

In this paper we specialize problem (1.1) in the following manner.

(1) We assume that $z_i = i, \ i = 0, ..., n$ and $f_0 = ... = f_{r-1} = 0, \ f_r = ... = f_n = 1$ for some r. The problem can be used in connection with arbitrary events $A_1, ..., A_n$, to obtain sharp lower and upper bounds for the probability of the union. In fact if we define $S_0 = 1$ and

$$S_k = \sum_{1 \leq i_1 < ... < i_k \leq n} P(A_{i_1} ... A_{i_k}), \quad k = 1, ..., n,$$

then by a well-known theorem (see, e.g., [11]) we have the equation

$$S_k = E\left[\binom{\xi}{k}\right], \quad k = 1, ..., n, \tag{1.2}$$

where ξ is the number of those events which occur. The equality constraints in (1.1) are just the same as $S_0 = 1$ and the equations in (1.2) for $k = 1, ..., m$ and the objective function is the probability of $\xi \geq r$ under the distribution $p_0, ..., p_n$. The distribution, however, is allowed to vary subject to the constraints, hence the optimum values of problem (1.1) provide us with the best possible bounds for the probability $P(\xi \geq r)$, given $S_1, ..., S_m$.

(2) We assume that $z_i = i, \ i = 0, ..., n$ and $f_r = 1, \ f_i = 0, \quad i \neq r$. In this case the optimum values of problem (1.1) provide us with the best possible lower and upper bounds for the probability $P(\xi = 1)$, given $S_1, ..., S_m$.

For small m values ($m \leq 4$) closed form bounds are presented in the literature. For power moment bounds see [10, 11]. Bounds for the probability of the union have been obtained by Fréchet [3] when $m = 1$, Dawson and Sankoff [2] when $m = 2$, Kwerel [5] when $m \leq 3$, Boros and Prékopa [1] when $m \leq 4$. In the last two paper bounds for the probability that at least r events occur, are also presented. For other closed form probability bounds see [4, 11]. In [7, 8, 9, 10, 11] Prékopa discovered that the probability bounds based on the binomial and power moments of the number of events that occur, out of a given collection $A_1, ..., A_n$, can be obtained as optimum values of discrete moment problems (DMP) and showed that for arbitrary m values simple dual algorithms solve problem (1.1) if f is of type (1) or (2). In [13], authors gave closed form formulas for expectations of convex functions of discrete random variables and the probability that at least 1 out of n events occurs when $m = 2$.

In this paper we formulate and use the binomial moment problem with finite, preassigned support and with given shape of the probability distribution to obtain sharp lower and upper bounds for unknown probabilities. We assume that the probability distribution $\{p_i\}$ is either decreasing (Type 1) or increasing (Type 2) or unimodal with a known modus (Type 3). The reasoning goes along the lines presented in above cited papers by Prékopa.

In Section 2 some basic theorems are given. In Section 3 and 4 we use the dual feasible basis structure theorems in [7, 8, 9, 10] to obtain sharp bounds for $P(\xi \geq r)$ and $P(\xi = r)$ in case of problem (1.1), where the first two moments are known. In Section 5 we give numerical examples to compare the sharp bounds obtained by the original binomial moment problem and the sharp bounds obtained by the transformed problems: Type 1, Type 2 and Type 3. In Section 6 we present two examples for the application of our bounding technique, where shape information about the unknown probability distribution can be used.

2 Basic Theorems

Consider the following binomial moment problem:

$$min(max) \sum_{i=r}^{n} p_i$$

subject to

$$\sum_{i=0}^{n} p_i = 1$$

$$\sum_{i=1}^{n} i p_i = S_1$$

$$\sum_{i=2}^{n} \binom{i}{2} p_i = S_2$$

$$\vdots$$

$$\sum_{i=m}^{n} \binom{i}{m} p_i = S_m$$

$$p_0, ..., p_n \geq 0 \ . \tag{2.1}$$

The following theorem [7, 8, 9, 10] describes the dual feasible bases in problem (2.1).

Theorem 1. *Let $0 \leq r \leq n$. A basis in problem (2.1) is a dual feasible basis if and only if it has one of the following structures (in terms of the subscripts of the basic vectors).*

Minimization problem, $m+1$ even

- $r \notin I$,
- $\{0, i, i+1, ..., j, j+1, r-1, k, k+1, ..., t, t+1\}$ *if* $2 \leq r \leq n-1$,
- $\{i, i+1, ..., j, j+1, r-1, k, k+1, ..., t, t+1, n\}$ *if* $1 \leq r \leq n-2$,
- $\{0, 1, i, i+1, ..., j, j+1\}$ *if* $r = 0$,
- $\{i, i+1, ..., j, j+1, n-1, n\}$ *if* $r = n$.

Minimization problem, $m+1$ odd

- $r \notin I$,
- $\{0, i, i+1, ..., j, j+1, r-1, r, r+1, k, k+1, ..., t, t+1, n\}$ *if* $2 \leq r \leq n-2$,
- $\{i, i+1, ..., j, j+1, r-1, r, r+1, k, k+1, ..., t, t+1\}$ *if* $1 \leq r \leq n-1$,
- $\{0, 1, i, i+1, ..., j, j+1, n\}$ *if* $r = 0$,
- $\{0, i, i+1, ..., j, j+1, n-1, n\}$ *if* $r = n$.

Maximization problem, $m+1$ even

- $\{i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1, n\}$ *if* $0 \leq r \leq n-1$,
- $\{0, i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1\}$ *if* $1 \leq r \leq n$.

Maximization problem, $m+1$ odd

- $\{i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1\}$ *if* $0 \leq r \leq n$,

• $\{0, i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1, n\}$ *if* $1 \leq r \leq n-1$, *where in all parentheses the numbers are arranged according to increasing order. If* $n > m+2$, *then all bases for which* $r \notin I$, *are dual degenerate. The bases in all other cases are dual nondegenerate.*

Now we consider the following problem:

$$min(max) \ \{p_r\}$$

subject to

$$\sum_{i=0}^{n} p_i = 1$$

$$\sum_{i=1}^{n} i p_i = S_1$$

$$\sum_{i=2}^{n} \binom{i}{2} p_i = S_2$$

$$\vdots$$

$$\sum_{i=m}^{n} \binom{i}{m} p_i = S_m$$

$$p_0, ..., p_n \geq 0 \ . \tag{2.2}$$

The following theorem [7, 8, 9, 10] characterizes the dual feasible basis in (2.2).

Theorem 2. *Let* $1 \leq r \leq n$. *A basis in Problems (2.2) is a dual feasible basis if and only if it has one of the following structures (in terms of the subscripts of the basic vectors).*

Minimization problem, $m+1$ even

• $I \subset \{0, ..., r-1\}$ *if* $r \geq m+1$,

• $\{0, i, i+1, ..., j, j+1, r-1, k, k+1, ..., t, t+1\}$ *if* $2 \leq r \leq n-1$,

• $\{i, i+1, ..., j, j+1, r-1, k, k+1, ..., t, t+1, n\}$ *if* $1 \leq r \leq n$.

Minimization problem, $m+1$ odd

• $I \subset \{0, ..., r-1\}$ *if* $r \geq m+1$,

• $\{0, i, i+1, ..., j, j+1, r-1, k, k+1, ..., t, t+1, n\}$ *if* $2 \leq r \leq n$,

• $\{i, i+1, ..., j, j+1, r-1, k, k+1, ..., t, t+1\}$ *if* $1 \leq r \leq n-1$.

Maximization problem, $m+1$ even

• $I \subset \{r, ..., n\}$ *if* $n-r \geq m$,

• $\{i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1, n\}$ *if* $1 \leq r \leq n-1$,

- $\{0, i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1\}$ *if* $1 \le r \le n$.

Maximization problem, $m+1$ even

- $I \subset \{r, ..., n\}$ *if* $n - r \ge m$,
- $\{i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1, n\}$ *if* $1 \le r \le n$,
- $\{0, i, i+1, ..., j, j+1, r, k, k+1, ..., t, t+1\}$ *if* $1 \le r \le n$, *where in all parentheses the numbers are arranged in increasing order. Those bases for which* $I \subset \{0, ..., r-1\}$ $(I \subset \{r, ..., n\})$ *are dual nondegenerate in the minimization (maximization) problem if* $r > m+1$ $(n - r + 1 > m + 1)$. *The bases in all other cases are dual nondegenerate.*

3 Sharp bounds for the probability that at least r events occur

In this section we consider the binomial moment problem (1.1). We assume that the distribution is either decreasing or increasing or unimodal with a known and fixed modus. We give sharp lower and upper bounds for $P(\xi \ge r)$ in case of three problem types: the probabilities $p_0, ..., p_n$ are (1) decreasing, (2) increasing; (3) form a unimodal sequence.

We look at the special case, where

$$z_i = i, \quad i = 0, \ ..., \ n \ , \quad f_0 = ... = f_{r-1} = 0, \quad f_r = ... = f_n = 1 \ .$$

In case of $m = 2$ we give the sharp lower and upper bounds for the probability that at least r out of n events occur. We look at the problem (1.1) but the constraints are supplemented by shape constraints of the unknown probability distribution $p_0, ..., p_n$.

3.1 TYPE 1: $p_0 \ge ... \ge p_n$

We assume that the probabilities $p_0, ..., p_n$ are unknown but satisfy the above inequalities.

Let $m = 2$. If we introduce the variables

$$v_0 = p_0 - p_1, \ ... \ , \ v_{n-1} = p_{n-1} - p_n, \ v_n = p_n,$$

then problem (1.1) can be written as

$$min(max)\{v_r + 2v_{r+1} + ... + (n-r+1)v_n\}$$

subject to

$$v_0 + 2v_1 + 3v_2 + 4v_3 + ... + (n+1)v_n = 1$$

$$\binom{2}{2} v_1 + \binom{3}{2} v_2 + \binom{4}{2} v_3 + ... + \binom{n+1}{2} v_n = S_1$$

$$\binom{2}{2} v_2 + \left[\binom{2}{2} + \binom{3}{2}\right] v_3 + ... + \left[\binom{2}{2} + ... + \binom{n}{2}\right] v_n = S_2$$

$$v_0,\ ...,\ v_n \geq 0\ . \tag{3.1}$$

Taking into account the equation:

$$1 + \binom{3}{2} + ... + \binom{k}{2} = \frac{(k-1)k(k+1)}{6}\ ,$$

the problem is the same as the following:

$$min(max)\{v_r + 2v_{r+1} + ... + (n-r+1)v_n\}$$

subject to

$$\begin{aligned} \sum_{i=0}^{n} (i+1)v_i &= 1 \\ \sum_{i=0}^{n} (i+1)iv_i &= 2S_1 \\ \sum_{i=0}^{n} (i+1)i(i-1)v_i &= 6S_2 \\ v_0, ..., v_n &\geq 0\ . \end{aligned} \tag{3.2}$$

Problem (3.2) is equivalent to the following:

$$min(max)\{v_r + 2v_{r+1} + ... + (n-r+1)v_n\}$$

subject to

$$\begin{aligned} v_0 + 2v_1 + 3v_2 + 4v_3 + ... + (n+1)v_n &= 1 \\ 2v_1 + 6v_2 + 12v_3 + ... + (n+1)nv_n &= 2S_1 \\ 6v_2 + 24v_3 + ... + (n+1)n(n-1)v_n &= 6S_2 \\ v_0, ..., v_n &\geq 0\ . \end{aligned} \tag{3.3}$$

Let A be the coefficient matrix of the equality constraint in (3.3). By the use of Theorem 1, dual feasible bases in the minimization and the maximization problems in (3.3) are in the form

$$B_{min} = \begin{cases} (a_0, a_{r-1}, a_n) & \text{if } \ 2 \leq r \leq n \\ (a_{r-1}, a_i, a_{i+1}) & \text{if } \ 1 \leq r \leq n-1 \end{cases}$$

and

$$B_{max} = \begin{cases} (a_0, a_r, a_n) & \text{if } \ 1 \leq r \leq n-1 \\ (a_r, a_j, a_{j+1}) & \text{if } \ 1 \leq r \leq n \end{cases}\ ,$$

respectively, where $1 \le i \le n-1$ and $1 \le j \le n-1$.

Optimality Conditions for B_{min}

The basis $B_{min} = (a_0, a_{r-1}, a_n)$ is also primal feasible if

$$\frac{3S_2}{n-1} \le S_1 \le \frac{3S_2}{r-2} \quad \text{and} \quad 2(n+r-2)S_1 - 6S_2 \le n(r-1). \tag{3.4}$$

In this case the sharp lower bound for $P(\xi \ge r)$ is

$$\frac{6S_2 - 2(r-2)S_1}{n(n+1)} \; . \tag{3.5}$$

We have the following two cases for the primal feasibility of the basis $B_{min} = (a_{r-1}, a_i, a_{i+1})$:

Case 1. Let $r \le i \le n-1$. The basis $B_{min} = (a_{r-1}, a_i, a_{i+1})$ is primal feasible if

$$\begin{aligned} 2(i+r-1)S_1 - 6S_2 &\ge (r-1)(i+1) \\ 2(i+r-2)S_1 - 6S_2 &\le i(r-1) \\ 4iS_1 - 6S_2 &\le i(i+1). \end{aligned} \tag{3.6}$$

In this case the sharp lower bound for $P(\xi \ge r)$ is

$$\frac{2(2i+r)S_1 - 6S_2}{(i+1)(i+2)} - \frac{2(r-1)}{i+2} \; . \tag{3.7}$$

Case 2. Let $0 \le i \le r-3$. The basis $B_{min} = (a_{r-1}, a_i, a_{i+1})$ is primal feasible if

$$\begin{aligned} 2(i+r-1)S_1 - 6S_2 &\le (r-1)(i+1) \\ 2(i+r-2)S_1 - 6S_2 &\ge i(r-1) \\ 4iS_1 - 6S_2 &\le i(i+1), \end{aligned} \tag{3.8}$$

where $i \le r-3$.

We remark that in this case the lower bound for $P(\xi \ge r)$ is 0.

Optimality Conditions for B_{max}

In case of maximization problem in (3.3) the basis $B_{max} = (a_0, a_r, a_n)$ is also primal feasible if

$$\frac{3S_2}{n-1} \le S_1 \le \frac{3S_2}{r-2} \quad \text{and} \quad 2(n+r-1)S_1 - 6S_2 \le nr \ . \tag{3.9}$$

The sharp upper bound for $P(\xi \ge r)$ can be given as follows:

$$\frac{2(n^2+nr-r^3+r^2+r-1)S_1 - 6(n-r^2+1)S_2}{r(r+1)n(n+1)} \ . \tag{3.10}$$

The basis $B_{max} = (a_r, a_j, a_{j+1})$ is also primal feasible if j is determined by the following conditions:

Case 1. If $r+1 \le j \le n-1$,

$$\begin{aligned} 2(j+r)S_1 - 6S_2 &\ge r(j+1) \\ 2(j+r-1)S_1 - 6S_2 &\le rj \\ 4jS_1 - 6S_2 &\le j(j+1) \ . \end{aligned} \tag{3.11}$$

In this case the sharp upper bound for $P(\xi \ge r)$ is

$$\frac{j-2r^2+2}{(r+1)(j+2)} + \frac{2r(r+2j+1)S_1 - 6rS_2}{(r+1)(j+1)(j+2)} \ . \tag{3.12}$$

Case 2. If $0 \le j \le r-2$,

$$\begin{aligned} 2(j+r)S_1 - 6S_2 &\le r(j+1) \\ 2(j+r-1)S_1 - 6S_2 &\ge rj \\ 4jS_1 - 6S_2 &\le j(j+1) \ . \end{aligned} \tag{3.13}$$

In this case the sharp upper bound for $P(\xi \ge r)$ is

$$\frac{j(j+1)}{(r+1)(r-j)(r-j-1)} - \frac{4jS_1 - 6S_2}{(r+1)(r-j)(r-j-1)} \ . \tag{3.14}$$

3.2 TYPE 2: $p_0 \le ... \le p_n$

Now we assume that the probability distribution is increasing. Let us introduce the variables $v_0 = p_0,\ v_1 = p_1 - p_0,\ ...,\ v_n = p_n - p_{n-1}$. In this case problem (1.1) can be written as

$$min(max)\{(n-r+1)(v_0 + ... + v_r) + (n-r)v_{r+1} + ... + v_n\}$$

subject to

$$(n+1)v_0 + nv_1 + (n-1)v_2 + ... + v_n = 1$$

$$\binom{n+1}{2}(v_0+v_1)+\left[\binom{n+1}{2}-1\right]v_2+...+\left[\binom{n+1}{2}-\binom{n}{2}\right]v_n=S_1$$

$$\left[\binom{2}{2}+...+\binom{n}{2}\right](v_0+v_1+v_2)+\left[\binom{3}{2}+...+\binom{n}{2}\right]v_3+...+\binom{n}{2}v_n=S_2$$

$$v \geq 0\ . \tag{3.15}$$

Taking into account the equations:

$$\binom{n+1}{2}-\binom{i}{2}=\frac{(n+i)(n-i+1)}{2}, \qquad 2 \leq i \leq n$$

and

$$\binom{i}{2}+...+\binom{n}{2}=\frac{(n+1)n(n-1)-(i-2)(i-1)i}{6}, \qquad 2 \leq i \leq n$$

problem (3.15) can be written as

$$min(max)\{(n-r+1)(v_0+...+v_r)+(n-r)v_{r+1}+...+v_n\}$$

subject to

$$(n+1)v_0+nv_1+(n-1)v_2+...+v_n=1$$

$$(n+1)n(v_0+v_1)+(n+2)(n-1)v_2+...+(n+i)(n-i+1)v_i+...+2nv_n=2S_1$$

$$(n+1)n(n-1)(v_0+v_1+v_2)+...+[(n+1)n(n-1)-(i-2)(i-1)i]v_i+...+3n(n-1)v_n=6S_2$$

$$v_0,...,v_n \geq 0\ . \tag{3.16}$$

Let A be the coefficient matrix of the equality constraint in (3.16). By the use of Theorem 1, dual feasible bases in the minimization and the maximization problems in (3.16) are in the form

$$B_{min}=\begin{cases}(a_0,a_{r-1},a_n) & \text{if } 2 \leq r \leq n\\ (a_{r-1},a_i,a_{i+1}) & \text{if } 1 \leq r \leq n-1\end{cases}$$

and

$$B_{max}=\begin{cases}(a_0,a_r,a_n) & \text{if } 1 \leq r \leq n-1\\ (a_r,a_j,a_{j+1}) & \text{if } 1 \leq r \leq n\end{cases},$$

respectively, where $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$.

Optimality Conditions for B_{min}

The basis $B_{min}=(a_0,a_{r-1},a_n)$ is also primal feasible if

$$4(n-1)S_1-6S_2 \geq n(n-1)$$

$$2(n+r-3)S_1 - 6S_2 \le n(r-2) \tag{3.17}$$
$$2(2n+r-3)S_1 - 6S_2 \le n(n+2r-3)\ .$$

By the use of this basis the the lower bound for $P(\xi \ge r)$ can be found as

$$1 \ - \ \frac{r(n+2r-3)}{(n+1)(r-1)} \ + \ \frac{n(n-1)}{(r-1)(n-r+1)(n-r+2)}$$
$$- \ \frac{2(2n^3+r^3+3nr-3n^2r-5r^2+6r-2n)S_1 - 6((n-r)(n-r+1)-r)S_2}{n(n+1)(n-r+1)(n-r+2)}\ . \tag{3.18}$$

The basis $B_{min} = (a_{r-1}, a_i, a_{i+1})$ is also primal feasible if i is determined by the following inequalities:

Case 1. If $r \le i \le n-1$, then

$$2(n+2i-1)S_1 - 6S_2 \le i(2n+i+1)$$
$$2(n+r+i-2)S_1 - 6S_2 \ge (r-1)(n+i+1) + ni \tag{3.19}$$
$$2(n+r+i-3)S_1 - 6S_2 \le (r-1)(n+i) + n(i-1)\ .$$

In this case the lower bound for $P(\xi \ge r)$ is

$$1 + \frac{2(n+2i-1)S_1 - 6S_2 - i(2n+j+1)}{(i-r+1)(i-r+2)(n-r+2)}\ . \tag{3.20}$$

Case 2. If $0 \le i \le r-3$, then

$$2(n+2i-1)S_1 - 6S_2 \le i(2n+i+1)$$
$$2(n+r+i-2)S_1 - 6S_2 \le (r-1)(n+i+1) + ni \tag{3.21}$$
$$2(n+r+i-3)S_1 - 6S_2 \ge (r-1)(n+i) + n(i-1)\ .$$

In this case the lower bound for $P(\xi \ge r)$ can be given as

$$\frac{i(i+2r-3) - 2(r+2i-3)S_1 + 6S_2}{(n-i)(n-i+1)(n-r+2)}\ . \tag{3.22}$$

Optimality Conditions for B_{max}

Now we give closed form formulas for the sharp upper bound for the probability that at least r events occur. First, we ensure the primal feasibility of the basis for maximization problem in (3.16).

$B_{max} = (a_0, a_r, a_n)$ is primal feasible if the following conditions are satisfied:

$$\begin{aligned} 4(n-1)S_1 - 6S_2 &\geq n(n-1) \\ 2(n+r-2)S_1 - 6S_2 &\leq n(r-1) \\ 2(2n+r-2)S_1 - 6S_2 &\leq n(n+2r-1)\ . \end{aligned} \tag{3.23}$$

We have the following upper bound for $P(\xi \geq r)$:

$$\frac{2r(2n+r-2)S_1 - 6S_2}{nr(n+1)} - \frac{2(r-1)}{n+1}\ . \tag{3.24}$$

The primal feasibility of $B_{max} = (a_r, a_j, a_{j+1})$ is ensured if j is determined by the following conditions:

Case 1. If $r+1 \leq j \leq n-1$,

$$\begin{aligned} 2(n+2j-1) - 6S_2 &\leq j(2n+j+1) \\ 2(n+j+r-1)S_1 - 6S_2 &\geq r(n+j+1) + nj \\ 2(n+j+r-2)S_1 - 6S_2 &\leq n(r+j-1) + rj\ . \end{aligned} \tag{3.25}$$

We remark that if $B_{max} = (a_r, a_j, a_{j+1})$, where $r+1 \leq j \leq n-1$ is the optimal basis, then the upper bound for $P(\xi \geq r)$ is equal to 1.

Case 2. If $0 \leq j \leq r-2$,

$$\begin{aligned} 2(n+2j-1) - 6S_2 &\leq j(2n+j+1) \\ 2(n+j+r-1)S_1 - 6S_2 &\leq r(n+j+1) + nj \\ 2(n+j+r-2)S_1 - 6S_2 &\geq n(r+j-1) + rj. \end{aligned} \tag{3.26}$$

In this case the upper bound for $P(\xi \geq r)$ is given as follows:

$$\frac{j(j+2r-1) - 2(r+2j-2)S_1 + 6S_2}{(n-j)(n-j+1)}\ . \tag{3.27}$$

3.3 TYPE 3: $p_0 \leq ... \leq p_k \geq ... \geq p_n$

In this section we assume that the distribution is unimodal with a known modus z_k. First we introduce the variables $v_i,\ i = 0, 1, ..., n$:

$$v_0 = p_0,\ v_1 = p_1 - p_0,\ \dots\ ,\ v_k = p_k - p_{k-1}\ ,$$

$$v_{k+1} = p_{k+1} - p_{k+2},\ \dots\ ,\ v_{n-1} = p_{n-1} - p_n,\ v_n = p_n\ .$$

We have the following two possibilities:

1. $0 \le r \le k$,

2. $k+1 \le r \le n$.

In the following two subsections we give sharp bound formulas for each case.

3.3.1 Case 1. $0 \le r \le k$

We assume that $0 \le r \le k$. In this case problem (1.1) can be written as

$$min(max)\{(k-r+1)(v_0+\dots+v_r)+(k-r)v_{r+1}+\dots+v_k+v_{k+1}+2v_{k+2}+\dots+(n-k)v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + \dots + v_k + v_{k+1} + \dots + (n-k)v_n = 1$$

$$\binom{k+1}{2}(v_0+v_1) + \left[\binom{k+1}{2} - \binom{i}{2}\right] v_i + \dots + kv_k + (k+1)v_{k+1}$$

$$+\left[\binom{k+3}{2} - \binom{k+1}{2}\right] v_{k+2} + \dots + \left[\binom{n+1}{2} - \binom{k+1}{2}\right] v_n = S_1$$

$$\left[\binom{2}{2} + \dots + \binom{k}{2}\right](v_0+v_1+v_2) + \left[\binom{3}{2} + \binom{k}{2}\right] v_3 + \binom{k}{2} v_k + \binom{k+1}{2} v_{k+1} +$$

$$\left[\binom{k+1}{2} + \binom{k+2}{2}\right] v_{k+1} + \dots + \left[\binom{k+1}{2} + \dots + \binom{n}{2}\right] v_n = S_2$$

$$v \ge 0\ . \qquad (3.28)$$

This is equivalent to

$$min(max)\{(k-r+1)(v_0+\dots+v_r)+(k-r)v_{r+1}+\dots+v_k+v_{k+1}+2v_{k+2}+\dots+(n-k)v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + \dots + v_k + v_{k+1} + \dots + (n-k)v_n = 1$$

$$(k+1)kv_0+(k+1)kv_1+\dots+(k+i)(k-i+1)v_i+\dots+2kv_k+2(k+1)v_{k+1}+\dots+(n-k)(n+k+1)v_n = 2S_1$$

$$(k+1)k(k-1)(v_0+v_1+v_2) + \dots + [(k^3-k)-(i-2)(i-1)i]v_i + \dots + 3k(k-1)v_k$$

$$+3(k+1)kv_{k+1} + \dots + (n-k)(n^2+nk+k^2-1)v_n = 6S_2$$

$$v \ge 0\ . \qquad (3.29)$$

Let A be the coefficient matrix of the equality constraints in (3.29). By Theorem 1, a dual feasible basis for minimization problem in (3.29) is of the form

$$B_{min} = \begin{cases} (a_0, a_{r-1}, a_n) & \text{if } \ 2 \le r \le n \\ (a_{r-1}, a_i, a_{i+1}) & \text{if } \ 1 \le r \le n-1 \end{cases},$$

where $1 \le i \le n-1$ and a dual feasible basis for maximization problem in (3.29) is of the form and

$$B_{max} = \begin{cases} (a_0, a_r, a_n) & \text{if } \ 1 \le r \le n-1 \\ (a_r, a_j, a_{j+1}) & \text{if } \ 1 \le r \le n \end{cases},$$

where and $1 \le j \le n-1$.

Optimality Conditions for B_{min}

The basis $B_{min} = (a_0, a_{r-1}, a_n)$ is also primal feasible if the following conditions are satisfied:

$$\begin{aligned} 4(k-1)S_1 - 6S_2 &\ge k(k-1) \\ 2(k+r-3)S_1 - 6S_2 &\le k(r-2) \\ 2(2k+r-3)S_1 - 6S_2 &\le k(k+2r-3) \ . \end{aligned} \tag{3.30}$$

We have the following lower bound for $P(\xi \ge r)$:

$$\begin{aligned} &1 \ - \ \frac{r(k+2r-3)}{(k+1)(r-1)} \ + \ \frac{k(k-1)}{(r-1)(k-r+1)(k-r+2)} \\ &- \frac{2(2k^3+r^3+3kr-3k^2r-5r^2+6r-2k)S_1 - 6((k-r)(k-r+1)-r)S_2}{k(k+1)(k-r+1)(k-r+2)} \ . \end{aligned} \tag{3.31}$$

The basis $B_{min} = (a_{r-1}, a_i, a_{i+1})$ is also primal feasible if i is determined by the following inequalities:

Case 1.1 If $0 \le i \le r-3$, then

$$\begin{aligned} 2(k+2i-1)S_1 - 6S_2 &\le i(2k+i+1) \\ 2(k+r+i-2)S_1 - 6S_2 &\le (r-1)(k+i+1) + ki \\ 2(k+r+i-3)S_1 - 6S_2 &\ge (r-1)(k+i) + k(i-1) \ . \end{aligned} \tag{3.32}$$

In this case the lower bound for $P(\xi \ge r)$ is

$$\frac{i(i+2r-3) - 2(r+2i-3)S_1 + 6S_2}{(k-i)(k-i+1)(k-r+2)} \ . \tag{3.33}$$

Case 1.2 If $r \le i \le n-1$, then

$$\begin{aligned} 2(k+2i-1)S_1 - 6S_2 &\le i(2k+i+1) \\ 2(k+r+i-2)S_1 - 6S_2 &\ge (r-1)(k+i+1)+ki \\ 2(k+r+i-3)S_1 - 6S_2 &\le (r-1)(k+i)+k(i-1)\ . \end{aligned} \tag{3.34}$$

The lower bound for $P(\xi \ge r)$ can be given as

$$1 + \frac{2(k+2i-1)S_1 - 6S_2 - i(2k+i+1)}{(i-r+1)(i-r+2)(k-r+2)}\ . \tag{3.35}$$

Case 1.3 If $k \le i \le n-1$, then

$$\begin{aligned} 2(2i+k+1)S_1 - 6S_2 &\le (i+1)(2k+i+2) \\ 2(r+i+k-1)S_1 - 6S_2 &\ge k(r+i)+(i+2)(r-1) \\ 2(r+i+k-2)S_1 - 6S_2 &\le ik+(i+k+1)(r-1) \end{aligned} \tag{3.36}$$

and the closed form formula for the lower bound for $P(\xi \ge r)$ is

$$1 + \frac{2(2i+k+1)S_1 - 6S_2 - (i+1)(2k+i+2)}{(i-r+2)(i-r+3)(k-r+2)}\ . \tag{3.37}$$

Case 1.4 If $i = k$, then

$$\begin{aligned} 2kS_1 - 2S_2 &\le k(k+1) \\ 2(2k+r-1)S_1 - 6S_2 &\ge (k+1)(k+2r-2) \\ 2(2k+r-3)S_1 - 6S_2 &\le k(k+2r-3)\ . \end{aligned} \tag{3.38}$$

In this case the lower bound for $P(\xi \ge r)$ is given by the following formula:

$$1 + \frac{6kS_1 - 6S_2 - 3k(k+1)}{(k-r+1)(k-r+2)(k-r+3)}\ . \tag{3.39}$$

Optimality Conditions for B_{max}

Now we give the primal feasibility conditions of a dual feasible basis for the maximization problem in (3.29).

$B_{max} = (a_0, a_r, a_n)$ is also primal feasible if

$$\begin{aligned} 2(n+k-1)S_1 - 6S_2 &\ge nk \\ 2(r+k-2)S_1 - 6S_2 &\le k(r-1) \end{aligned} \tag{3.40}$$

$$2(n+r+k-1)S_1 - 6S_2 \le r(n+k+1) + nk \ .$$

If the conditions in (3.40) are satisfied, then the upper bound for $P(\xi \ge r)$ can be obtained as

$$\frac{2(n+r+k-1)S_1 - 6S_2 - (n+k+1)(r-1)}{(n+1)(k+1)} \ . \tag{3.41}$$

The basis $B_{max} = (a_r, a_j, a_{j+1})$ is also primal feasible is j is determined by the following conditions:

$$\begin{aligned} 2(2j+k-1)S_1 - 6S_2 &\le j(j+2k+1) \\ 2(j+r+k-1)S_1 - 6S_2 &\le r(j+k+1) + jk \\ 2(j+r+k-2)S_1 - 6S_2 &\ge k(j+r-1) + rj \ , \end{aligned} \tag{3.42}$$

where $0 \le j \le r-2$.

In this case the upper bound for $P(\xi \ge r)$ can be given as

$$\frac{j(j+2r-1) - 2(2j+r-2)S_1 + 6S_2}{(k-j)(k-j+1)}. \tag{3.43}$$

The basis $B_{max} = (a_r, a_j, a_{j+1}), \ \ r+1 \le j \le k-2$ is also primal feasible if

$$\begin{aligned} 2(k+j+r-1)S_1 - 6S_2 &\ge r(k+j+1) + kj \\ 2(k+j+r-2)S_1 - 6S_2 &\le k(r+j-1) + rj \\ 2(k+2j-1)S_1 - 6S_2 &\le j(2k+j+1) \ . \end{aligned} \tag{3.44}$$

The basis $B_{max} = (a_r, a_j, a_{j+1}), \ \ k+1 \le j \le n-1$ is also primal feasible if

$$\begin{aligned} 2(rj+k+1)S_1 - 6S_2 &\le (j+1)(j+2k+2) \\ 2(j+k+r)S_1 - 6S_2 &\ge r(j+k+2) + k(j+1) \\ 2(j+r+k-1)S_1 - 6S_2 &\le r(j+k+2) + j(k-1) - 2 \ . \end{aligned} \tag{3.45}$$

The basis $B_{max} = (a_r, a_k, a_{k+1})$ is also primal feasible if

$$\begin{aligned} 2kS_1 - 2S_2 &\le k(k+1) \\ 2(2k+r)S_1 - 6S_2 &\ge (k+1)(2r+k) \\ 2(2k+r-2)S_1 - 6S_2 &\le k(k+2r-1) \ . \end{aligned} \tag{3.46}$$

We remark that if $r+1 \le j \le k-2$ or $k+1 \le j \le n-1$ or $j = k$, then the optimum value of the maximization problem in (3.29) is 1. Thus, the upper bound for $P(\xi \ge r)$ is 1.

3.3.2 Case 2. $k+1 \le r \le n$

Now we assume that $k+1 \le r \le n$. Taking this into account problem (1.1) can be written as follows:

$$min(max)\{v_r + 2v_{r+1} + ... + (n-r+1)v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$\binom{k+1}{2}(v_0+v_1) + \left[\binom{k+1}{2} - \binom{i}{2}\right] v_i + ... + kv_k + (k+1)v_{k+1}$$

$$+ \left[\binom{k+3}{2} - \binom{k+1}{2}\right] v_{k+2} + ... + \left[\binom{n+1}{2} - \binom{k+1}{2}\right] v_n = S_1$$

$$\left[\binom{2}{2} + ... + \binom{k}{2}\right](v_0+v_1+v_2) + \left[\binom{3}{2} + \binom{k}{2}\right] v_3 + \binom{k}{2} v_k + \binom{k+1}{2} v_{k+1} +$$

$$\left[\binom{k+1}{2} + \binom{k+2}{2}\right] v_{k+1} + ... + \left[\binom{k+1}{2} + ... + \binom{n}{2}\right] v_n = S_2$$

$$v \ge 0 \ . \qquad (3.47)$$

Similarly, this is equivalent to

$$min(max)\{v_r + 2v_{r+1} + ... + (n-r+1)v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$(k+1)kv_0 + (k+1)kv_1 + ... + (k+i)(k-i+1)v_i + ... + 2kv_k + 2(k+1)v_{k+1} + ... + (n-k)(n+k+1)v_n = 2S_1$$

$$(k+1)k(k-1)(v_0+v_1+v_2) + ... + [(k^3-k) - (i-2)(i-1)i]v_i + ... + 3k(k-1)v_k$$

$$+3(k+1)kv_{k+1} + ... + (n-k)(n^2+nk+k^2-1)v_n = 6S_2$$

$$v \ge 0 \ . \qquad (3.48)$$

Let A be the coefficient matrix of the equality constraints in (3.41). By Theorem 1, a dual feasible basis for minimization problem in (3.41) is of the form

$$B_{min} = \begin{cases} (a_0, a_{r-1}, a_n) & \text{if } 2 \le r \le n \\ (a_{r-1}, a_i, a_{i+1}) & \text{if } 1 \le r \le n-1 \end{cases},$$

where $1 \le i \le n-1$ and a dual feasible basis for maximization problem in (3.29) is of the form and

$$B_{max} = \begin{cases} (a_0, a_r, a_n) & \text{if } 1 \le r \le n-1 \\ (a_r, a_j, a_{j+1}) & \text{if } 1 \le r \le n \end{cases},$$

where and $1 \leq j \leq n-1$.

Optimality Conditions for B_{min}

The optimality of the basis $B_{min} = (a_0, a_{r-1}, a_n)$ is ensured if the following conditions are satisfied:

$$\begin{aligned} 2(n+k-1)S_1 - 6S_2 &\geq nk \\ 2(r+k-2)S_1 - 6S_2 &\leq k(r-1) \\ 2(n+r+k-1)S_1 - 6S_2 &\leq r(n+1) + k(n-r) \ . \end{aligned} \tag{3.49}$$

If the basis $B_{min} = (a_0, a_{r-1}, a_n)$ is optimal,then the lower bound for $P(\xi \geq r)$ can be obtained as

$$\frac{k(r-1) - 2(r+k-2)S_1 + 6S_2}{(n+1)(n-k)} \ . \tag{3.50}$$

The basis $B_{min} = (a_{r-1}, a_i, a_{i+1})$ is also primal feasible if i is determined by the following inequalities:

Case 2.1 If $0 \leq i \leq k-1$ or $k+1 \leq i \leq r-3$, then the optimum value of the minimization problem in (3.41) is 0. Thus, the lower bound for $P(\xi \geq r)$ is 0.

Case 2.2 Let $r \leq i \leq n-1$.

$$\begin{aligned} 2(r+i+k)S_1 - 6S_2 &\geq r(i+k+2) + k(i+1) \\ 2(r+i+k-1)S_1 - 6S_2 &\leq r(i+k) + ik \\ 2(2i+k+1)S_1 - 6S_2 &\leq (i+1)(i+2k+2) + r(i-r+1) \ . \end{aligned} \tag{3.51}$$

In this case the following is the lower bound for $P(\xi \geq r)$:

$$\frac{2(r+2i)S_1 - 6S_2 - k(2i-k+1) - r(3i-k+2)}{(i-k)(i-k+1)} \ . \tag{3.52}$$

Optimality Conditions for B_{max}

Now we give optimality conditions of a dual feasible basis of the maximization problem in (3.41).

The basis $B_{max} = (a_0, a_r, a_n)$ is optimal if

$$2(r+k-1)S_1 - 6S_2 \leq rk$$

$$2(n+r+k)S_1 - 6S_2 \le (n+1)(r+k+1) + rk \tag{3.53}$$
$$2(n+k-1)S_1 - 6S_2 \ge nk \ .$$

The upper bound for $P(\xi \ge r)$ is

$$\frac{2(n(n+r)-(r-1)^2(r+1)+rk(k-1))S_1 - 6(n+1-r(r-k))S_2}{(n-k)(r-k)(n+1)(r+1)}$$
$$-\frac{n(n+1)-(r-k)(n-r^2+1)}{(n-k)(r-k)(n+1)(r+1)} \ . \tag{3.54}$$

The basis $B_{max} = (a_r, a_j, a_{j+1})$ is also primal feasible if the following conditions are satisfied:

Case 2.1 If $0 \le j \le k-1$, then

$$\begin{aligned} 2(r+j+k)S_1 - 6S_2 &\le (j+1)(r+k+1) + rk \\ 2(r+j+k-1)S_1 - 6S_2 &\ge j(r+k+1) + rk \\ 2(2j+k-1)S_1 - 6S_2 &\le j(j+2k+1) \ . \end{aligned} \tag{3.55}$$

In this case the upper bound for $P(\xi \ge r)$ is given as

$$\frac{j(j+2k+1) - 2(2j+k-1)S_1 + 6S_2}{(r-k)(r-j)(r-j+1)} \ . \tag{3.56}$$

Case 2.2 Let $k+1 \le j \le r-2$. In this case the basis $B_{max} = (a_r, a_j, a_{j+1})$ is optimal if

$$\begin{aligned} 2(r+j+k)S_1 - 6S_2 &\le (j+1)(r+k+1) + rk \\ 2(r+j+k+1)S_1 - 6S_2 &\le (j+2)(r+k+1) + rk \\ 2(2j+k+1)S_1 - 6S_2 &\le (j+2)(r+k+1) + rk \end{aligned} \tag{3.57}$$

and the upper bound for $P(\xi \ge r)$ is

$$\frac{(j+1)(j+2k+2) - 2(2j+k+1)S_1 + 6S_2}{(r-k)(r-j)(r-j-1)} \ . \tag{3.58}$$

Case 2.3 If $r+1 \le j \le n-1$, then

$$\begin{aligned} 2(r+j+k)S_1 - 6S_2 &\le (j+1)(r+k+1) + rk \\ 2(r+j+k+1)S_1 - 6S_2 &\ge (j+2)(r+k+1) + rk \\ 2(2j+k+1)S_1 - 6S_2 &\le (j+1)(j+2k+2) \ . \end{aligned} \tag{3.59}$$

The closed form formula for the upper bound for $P(\xi \geq r)$ can be obtained as follows:

$$\frac{1}{r-k} - \frac{(2j-k+2)(r+k+1)+rk}{(r-k)(j-k)(j-k+1)} + \frac{2(2j+r+1)(r-k-1)S_1 - 6(r-k-1)S_2}{(r-k)(j-k)(j-k+1)} . \tag{3.60}$$

4 Sharp bounds for the probability that exactly r events occur

In this section we consider the special case of the binomial moment problem (1.1), where

$$z_i = i, \quad i = 0,\ ...,\ n\ , \quad f_r = 1, \quad f_j = 0,\ j \neq r.$$

In case of $m = 2$ we give the sharp lower and upper bounds for the probability that exactly r out of n events occur. We look at the problem (1.1) but the constraints are supplemented by shape constraints of the unknown probability distribution $p_0, ..., p_n$.

In the following three subsections we use the same shape constraints that we have used in Section 3.1-3.3.

4.1 TYPE 1: $p_0 \geq ... \geq p_n$

Let $m = 2$. If we introduce the variables

$$v_0 = p_0 - p_1,\ ...\ ,\ v_{n-1} = p_{n-1} - p_n,\ v_n = p_n,$$

then problem (1.1) can be written as

$$\begin{gathered} min(max)\{v_r + v_{r+1} + ... + v_n\} \\ \text{subject to} \\ v_0 + 2v_1 + 3v_2 + 4v_3 + ... + (n+1)v_n = 1 \\ \binom{2}{2} v_1 + \binom{3}{2} v_2 + \binom{4}{2} v_3 + ... + \binom{n+1}{2} v_n = S_1 \\ \binom{2}{2} v_2 + \left[\binom{2}{2} + \binom{3}{2}\right] v_3 + ... + \left[\binom{2}{2} + ... + \binom{n}{2}\right] v_n = S_2 \\ v_0,\ ...,\ v_n \geq 0\ . \end{gathered} \tag{4.1}$$

Taking into account the equation:

$$1 + \binom{3}{2} + ... + \binom{k}{2} = \frac{(k-1)k(k+1)}{6} \ ,$$

the problem is the same as the following:

$$min(max)\{v_r + v_{r+1} + ... + v_n\}$$

subject to

$$\begin{aligned} \sum_{i=0}^{n} (i+1)v_i &= 1 \\ \sum_{i=0}^{n} (i+1)iv_i &= 2S_1 \\ \sum_{i=0}^{n} (i+1)i(i-1)v_i &= 6S_2 \\ v_0, ..., v_n &\geq 0 \ . \end{aligned} \tag{4.2}$$

Problem (4.2) is equivalent to the following:

$$min(max)\{v_r + v_{r+1} + ... + v_n\}$$

subject to

$$\begin{aligned} v_0 + 2v_1 + 3v_2 + 4v_3 + ... + (n+1)v_n &= 1 \\ 2v_1 + 6v_2 + 12v_3 + ... + (n+1)nv_n &= 2S_1 \\ 6v_2 + 24v_3 + ... + (n+1)n(n-1)v_n &= 6S_2 \\ v_0, ..., v_n &\geq 0 \ . \end{aligned} \tag{4.3}$$

Let A be the coefficient matrix of the equality constraint in (4.3). By the use of Theorem 2, dual feasible bases in the minimization and the maximization problems in (4.3) are in the form

$$B_{min} = \begin{cases} (a_{r-1}, a_r, a_{r+1}) & \text{if } 1 \leq r \leq n-1 \\ (a_0, a_1, a_n) & \text{if } r = 0 \\ (a_0, a_{n-1}, a_n) & \text{if } r = n \end{cases}$$

and

$$B_{max} = \begin{cases} (a_0, a_r, a_n) & \text{if } 1 \leq r \leq n-1 \\ (a_r, a_j, a_{j+1}) & \text{if } 0 \leq r \leq n \end{cases} \ ,$$

respectively, where $1 \leq j \leq n-1$.

Optimality Conditions for B_{min}

The basis $B_{min} = (a_{r-1}, a_r, a_{r+1})$ is optimal if

$$\begin{aligned} 2(2r-1)S_1 - 6S_2 &\geq (r-1)(r+1) \\ 4(r-1)S_1 - 6S_2 &\leq r(r-1) \\ 4rS_1 - 6S_2 &\leq r(r+1)\ . \end{aligned} \tag{4.4}$$

If (4.4) is satisfied, then the lower bound for $P(\xi = r)$ can be given as

$$\frac{4(r^2+3r-1)S_1 - 6(r+3)S_2}{2(r+1)(r+2)} - \frac{(r-1)(r+4)}{2(r+2)}\ . \tag{4.5}$$

The basis $B_{min} = (a_0, a_1, a_n)$ is optimal if

$$S_1 \geq \frac{3S_2}{n-1} \quad \text{and} \quad 2nS_1 - 6S_2 \leq n\ . \tag{4.6}$$

In this case the lower bound for $P(\xi = r)$ can be obtained as

$$1 - S_1 + \frac{3S_2}{n+1}\ . \tag{4.7}$$

The basis $B_{min} = (a_0, a_{n-1}, a_n)$ is optimal if

$$\frac{3S_2}{n-1} \leq S_1 \leq \frac{3S_2}{n-2} \quad \text{and} \quad 4(n-1)S_1 - 6S_2 \leq n(n-1)\ . \tag{4.8}$$

The lower bound for $P(\xi = r)$ is given as follows:

$$\frac{6S_2 - 2(n-2)S_1}{n(n+1)}\ . \tag{4.9}$$

Optimality Conditions for B_{max}

First we remark that dual feasible bases of the maximization problem in (4.3) are of the same type as the dual feasible bases of the maximization problem in (3.3). Therefore we will obtain the same optimality conditions in Section 3.1. However, since the objective functions in (3.3) and (4.3) are different we obtain different optimum values for those problems.

If the basis $B_{max} = (a_0, a_r, a_n)$ is also primal feasible, i.e., (3.9) is satisfied, then the upper bound for $P(\xi = r)$ can be given as

$$\frac{2(n^2+nr+r^2-1)S_1 - 6(n+r+1)S_2}{r(r+1)n(n+1)}. \tag{4.10}$$

If the basis $B_{max} = (a_r, a_j, a_{j+1}),\quad 0 \le j \le r-2$ is also primal feasible, i.e., (3.11) is satisfied, then the upper bound for $P(\xi = r)$ can be given as

$$\frac{j(j+1) - 4jS_1 + 6S_2}{(r+1)(r-j)(r-j-2)}\ . \tag{4.11}$$

If the basis $B_{max} = (a_r, a_j, a_{j+1}),\quad r+1 \le j \le n-1$ is also primal feasible, i.e., (3.13) is satisfied, then the upper bound for $P(\xi = r)$ can be given as

$$\frac{j+2r+2}{(r+1)(j+2)} - \frac{2(2j+r+1)S_1 - 6S_2}{(r+1)(j+1)(j+2)}\ . \tag{4.12}$$

4.2 TYPE 2: $p_0 \le ... \le p_n$

Let us introduce the variables $v_0 = p_0,\ v_1 = p_1 - p_0,\ ...,\ v_n = p_n - p_{n-1}$. In this case problem (1.1) can be written as

$$min(max)\{v_0 + ... + v_r\}$$

subject to

$$(n+1)v_0 + nv_1 + (n-1)v_2 + ... + v_n = 1$$

$$\binom{n+1}{2}(v_0+v_1) + \left[\binom{n+1}{2} - 1\right]v_2 + ... + \left[\binom{n+1}{2} - \binom{n}{2}\right]v_n = S_1$$

$$\left[\binom{2}{2} + ... + \binom{n}{2}\right](v_0+v_1+v_2) + \left[\binom{3}{2} + ... + \binom{n}{2}\right]v_3 + ... + \binom{n}{2}v_n = S_2$$

$$v \ge 0\ . \tag{4.13}$$

Taking into account the equations:

$$\binom{n+1}{2} - \binom{i}{2} = \frac{(n+i)(n-i+1)}{2}, \qquad 2 \le i \le n$$

and

$$\binom{i}{2} + ... + \binom{n}{2} = \frac{(n+1)n(n-1) - (i-2)(i-1)i}{6}, \qquad 2 \le i \le n$$

problem (4.13) can be written as

$$min(max)\{v_0 + ... + v_r\}$$

subject to

$$(n+1)v_0 + nv_1 + (n-1)v_2 + ... + v_n = 1$$

$$(n+1)n(v_0+v_1)+(n+2)(n-1)v_2+...+(n+i)(n-i+1)v_i+...+2nv_n=2S_1$$
$$(n+1)n(n-1)(v_0+v_1+v_2)+...+[(n+1)n(n-1)-(i-2)(i-1)i]v_i+...+3n(n-1)v_n=6S_2$$
$$v_0,...,v_n \geq 0 \; . \tag{4.14}$$

Let A be the coefficient matrix of the equality constraint in (4.14). By the use of Theorem 1, dual feasible bases in the minimization and the maximization problems in (4.14) are in the form

$$B_{min} = \begin{cases} (a_{r-1}, a_r, a_{r+1}) & \text{if } \; 1 \leq r \leq n-1 \\ (a_0, a_1, a_n) & \text{if } \; r=0 \\ (a_0, a_{n-1}, a_n) & \text{if } \; r=n \end{cases}$$

and

$$B_{max} = \begin{cases} (a_0, a_r, a_n) & \text{if } \; 1 \leq r \leq n-1 \\ (a_r, a_j, a_{j+1}) & \text{if } \; 0 \leq r \leq n \end{cases} ,$$

respectively, where $1 \leq j \leq n-1$.

Optimality Conditions for B_{min}

$B_{min} = (a_{r-1}, a_r, a_{r+1})$ is optimal if

$$\begin{aligned} 2(n+2r-1)S_1 - 6S_2 &\leq r(2n+r+1) \\ 2(n+2r-2)S_1 - 6S_2 &\geq r(2n-r)+n+1 \\ 2(n+2r-3)S_1 - 6S_2 &\leq (r-1)(2n+r) \; . \end{aligned} \tag{4.15}$$

If (4.15) is satisfied, then the lower bound for $P(\xi = r)$ is the following:

$$\frac{2(n(n-r)+(r-1)(2r-7))S_1 - 6(n-r+3)S_2}{2(n-r+1)(n-r+2)}$$
$$+\frac{r(2n+r+1)}{2(n-r+2)} + \frac{n-2nr-r^2+1}{n-r+1} \; . \tag{4.16}$$

The basis $B_{min} = (a_0, a_1, a_n)$ is optimal if

$$\begin{aligned} 4(n-1)S_1 - 6S_2 &\geq n(n-1) \\ S_1 &\leq \frac{3S_2}{n-1} \\ 2(2n-1)S_1 - 6S_2 &\leq n(n+1) \; . \end{aligned} \tag{4.17}$$

If (4.17) is satisfied, then the lower bound for $P(\xi = r)$ is the following:

$$1 - \frac{2(2n-1)S_1 - 6S_2}{n(n+1)} \; . \tag{4.18}$$

The basis $B_{min} = (a_0, a_{n-1}, a_n)$ is optimal if

$$\begin{aligned} 4(n-1)S_1 - 6S_2 &\geq n(n-1) \\ 2(2n-3)S_1 - 6S_2 &\leq n(n-2) \\ 2(n-1)S_1 - 2S_2 &\leq n(n-1) \ . \end{aligned} \tag{4.19}$$

If (4.19) is satisfied, then the lower bound for $P(\xi = r)$ is the following:

$$\frac{(n-1)(n-2) - 4(n-2)S_1 + 6S_2}{2(n+1)} \ . \tag{4.20}$$

Optimality Conditions for B_{max}

Since dual feasible bases of the maximization problem in (4.14) are of the same type as the dual feasible bases of the maximization problem in (3.16), we give only the upper bound formulas for $P(\xi = r)$.

If the basis $B_{max} = (a_0, a_r, a_n)$ is also primal feasible, i.e., (3.23) is satisfied, then the upper bound for $P(\xi = r)$ can be given as

$$\frac{(3n(n-1) - (r-1)(r-2))S_1 - 6(2n-r+1)S_2}{n(n+1)(n-r)(n-r+1)} - \frac{(r-1)(3n-2r+1)}{(n+1)(n-2)(n-r+1)} \ . \tag{4.21}$$

If the basis $B_{max} = (a_r, a_j, a_{j+1}), \quad 0 \leq j \leq r-2$ is also primal feasible, i.e., (3.11) is satisfied, then the upper bound for $P(\xi = r)$ can be given as

$$\frac{j(j+2r-1) - 2(r+2j-2)S_1 + 6S_2}{(n-j)(n-j+1)(n-r+1)} \ . \tag{4.22}$$

If the basis $B_{max} = (a_r, a_j, a_{j+1}), \quad r+1 \leq j \leq n-1$ is also primal feasible, i.e., (3.13) is satisfied, then the upper bound for $P(\xi = r)$ can be given as

$$\frac{j(2n+j+1) - 2(n+2j-1)S_1 + 6S_2}{(j-r)(j-r+1)(n-r+1)} \ . \tag{4.23}$$

4.3 TYPE 3: $p_0 \leq ... \leq p_k \geq ... \geq p_n$

Now we assume that the distribution is unimodal with a known modus.

We introduce the variables $v_i, \ i = 0, 1, ..., n$:

$$v_0 = p_0, \ v_1 = p_1 - p_0, \ \dots \ , \ v_k = p_k - p_{k-1} \ ,$$

$$v_{k+1} = p_{k+1} - p_{k+2}, \ \dots \ , \ v_{n-1} = p_{n-1} - p_n, \ v_n = p_n \ .$$

We have the following two possibilities:

1. $0 \leq r \leq k$,

2. $k+1 \leq r \leq n$.

4.3.1 Case 1.

We assume that $0 \le r \le k$. In this case problem (1.1) can be written as

$$min(max)\{v_0 + ... + v_r\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$\binom{k+1}{2}(v_0+v_1) + \left[\binom{k+1}{2} - \binom{i}{2}\right] v_i + ... + kv_k + (k+1)v_{k+1}$$

$$+ \left[\binom{k+3}{2} - \binom{k+1}{2}\right] v_{k+2} + ... + \left[\binom{n+1}{2} - \binom{k+1}{2}\right] v_n = S_1$$

$$\left[\binom{2}{2} + ... + \binom{k}{2}\right](v_0+v_1+v_2) + \left[\binom{3}{2} + \binom{k}{2}\right] v_3 + \binom{k}{2} v_k + \binom{k+1}{2} v_{k+1} +$$

$$\left[\binom{k+1}{2} + \binom{k+2}{2}\right] v_{k+1} + ... + \left[\binom{k+1}{2} + ... + \binom{n}{2}\right] v_n = S_2$$

$$v \ge 0 \ . \tag{4.24}$$

This is equivalent to

$$min(max)\{v_0 + ... + v_r\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$(k+1)kv_0 + (k+1)kv_1 + ... + (k+i)(k-i+1)v_i + ... + 2kv_k + 2(k+1)v_{k+1} + ... + (n-k)(n+k+1)v_n = 2S_1$$

$$(k+1)k(k-1)(v_0+v_1+v_2) + ... + [(k^3-k) - (i-2)(i-1)i]v_i + ... + 3k(k-1)v_k$$

$$+3(k+1)kv_{k+1} + ... + (n-k)(n^2+nk+k^2-1)v_n = 6S_2$$

$$v \ge 0 \ . \tag{4.25}$$

Let A be the coefficient matrix of the equality constraints in (4.25). By Theorem 2, a dual feasible basis for minimization problem in (4.25) is of the form

$$B_{min} = \begin{cases} (a_{r-1}, a_r, a_{r+1}) & \text{if } \ 1 \le r \le n-1 \\ (a_0, a_1, a_n) & \text{if } \ r = 0 \\ (a_0, a_{n-1}, a_n) & \text{if } \ r = n \end{cases}$$

and a dual feasible basis for maximization problem in (3.29) is of the form and

$$B_{max} = \begin{cases} (a_0, a_r, a_n) & \text{if } \ 1 \le r \le n-1 \\ (a_r, a_j, a_{j+1}) & \text{if } \ 0 \le r \le n \end{cases} ,$$

where $1 \le j \le n-1$.

Optimality Conditions for B_{min}

First we remark that the basis $B_{min} = (a_0, a_{n-1}, a_n)$, $r = n$ is ruled out because it contradicts with the assumption that $r \le k$.

The basis $B_{min} = (a_{r-1}, a_r, a_{r+1})$ is also primal feasible if the following conditions are satisfied:

Case 1.1 If $0 \le r \le k-1$, then

$$\begin{aligned} 2(k+2r-1)S_1 - 6S_2 &\le r(2k+r+1) \\ 2(k+2r-2)S_1 - 6S_2 &\ge r(2k-r)+k+1 \\ 2(k+2r-3)S_1 - 6S_2 &\le (r-1)(2k+r)\ . \end{aligned} \tag{4.26}$$

If (4.15) is satisfied, then the lower bound for $P(\xi = r)$ is the following:

$$\frac{2(k(k-r)+(r-1)(2r-7))S_1 - 6(k-r+3)S_2}{2(k-r+1)(k-r+2)} + \frac{r(2k+r+1)}{2(k-r+2)} + \frac{k-2kr-r^2+1}{k-r+1}\ . \tag{4.27}$$

Case 1.2 If $r = k$, then

$$\begin{aligned} 2(3k-1)S_1 - 6S_2 &\ge (3k-2)(k+1) \\ 2(k-1)S_1 - 2S_2 &\le k(k-1) \\ 2kS_1 - 2S_2 &\le k(k+1)\ . \end{aligned} \tag{4.28}$$

In this case the lower bound for $P(\xi = r)$ can be given as follows:

$$(2k-1)S_1 - 2S_2 - (k-1)(k+1)\ . \tag{4.29}$$

The basis $B_{min} = (a_0, a_1, a_n)$ is also primal feasible if

$$\begin{aligned} 2(n+k-1)S_1 - 6S_2 &\ge nk \\ S_1 &\le \frac{3S_2}{k-1} \\ 2(n(n+1)+k-1)S_1 - 6S_2 &\le n(n+1)(k+1)\ . \end{aligned} \tag{4.30}$$

If (4.30) is satisfied, the lower bound for $P(\xi = r)$ is the following:

$$\frac{2(n^2+nk^2+k-1)S_1-6(nk+n+1)S_2}{k(k+1)n(n+1)}\ . \tag{4.31}$$

Optimality Conditions for B_{max}

Since dual feasible bases of the maximization problem in (4.25) are of the same type as the dual feasible bases of the maximization problem in (3.29), we give only the upper bound formulas for $P(\xi = r)$.

If $B_{max} = (a_0, a_r, a_n)$ is also primal feasible, i.e., (3.40) is satisfied, then the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{nr+nk+rk+r}{r(k+1)(n+1)} - \frac{nk}{r(k-r+1)(n-r+1)}$$
$$+\ \frac{2(n^2-r^2+3r-3+nk-k^2)S_1-6(n+k-r-2)S_2}{(k+1)(n+1)(k-r+1)(n-r+1)}\ . \tag{4.32}$$

If $B_{max} = (a_r, a_j, a_{j+1}),\ \ 0 \le j \le r-2$ is also primal feasible, i.e., (3.42) is satisfied, then the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{j(j+2r-1)-2(r+2j-2)S_1+6S_2}{(k-j)(k-j+1)(k-r+1)}\ . \tag{4.33}$$

$B_{max} = (a_r, a_j, a_{j+1}),\ \ r+1 \le j \le k-2$ is also primal feasible if j is determined by (3.44). In this case the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{j(2k+j+1)-2(k+2j-1)S_1+6S_2}{(j-r)(j-r+1)(k-r+1)}\ . \tag{4.34}$$

The basis $B_{max} = (a_r, a_j, a_{j+1}),\ \ k+1 \le j \le n-1$ is also primal feasible if (3.45) is satisfied. In this case the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{(j+1)(j+2k+2)-2(2j+k+1)S_1+6S_2}{(j-r+1)(j-r+2)(k-r+1)}\ . \tag{4.35}$$

The basis $B_{max} = (a_r, a_k, a_{k+1})$ is also primal feasible if (3.46) is satisfied. In this case the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{3k(k+1)-6kS_1+6S_2}{(k-r)(k-r+1)(k-r+2)}\ . \tag{4.36}$$

4.3.2 Case 2.

We assume that $k+1 \le r \le n-1$. In this case problem (1.1) can be written as

$$min(max)\{v_r + ... + v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$\left(\begin{array}{c} k+1 \\ 2 \end{array}\right)(v_0+v_1) + \left[\left(\begin{array}{c} k+1 \\ 2 \end{array}\right) - \left(\begin{array}{c} i \\ 2 \end{array}\right)\right] v_i + ... + kv_k + (k+1)v_{k+1}$$

$$+\left[\left(\begin{array}{c} k+3 \\ 2 \end{array}\right) - \left(\begin{array}{c} k+1 \\ 2 \end{array}\right)\right] v_{k+2} + ... + \left[\left(\begin{array}{c} n+1 \\ 2 \end{array}\right) - \left(\begin{array}{c} k+1 \\ 2 \end{array}\right)\right] v_n = S_1$$

$$\left[\left(\begin{array}{c} 2 \\ 2 \end{array}\right) + ... + \left(\begin{array}{c} k \\ 2 \end{array}\right)\right](v_0+v_1+v_2) + \left[\left(\begin{array}{c} 3 \\ 2 \end{array}\right) + \left(\begin{array}{c} k \\ 2 \end{array}\right)\right] v_3 + \left(\begin{array}{c} k \\ 2 \end{array}\right) v_k + \left(\begin{array}{c} k+1 \\ 2 \end{array}\right) v_{k+1} +$$

$$\left[\left(\begin{array}{c} k+1 \\ 2 \end{array}\right) + \left(\begin{array}{c} k+2 \\ 2 \end{array}\right)\right] v_{k+1} + ... + \left[\left(\begin{array}{c} k+1 \\ 2 \end{array}\right) + ... + \left(\begin{array}{c} n \\ 2 \end{array}\right)\right] v_n = S_2$$

$$v \ge 0 \ . \tag{4.37}$$

This is equivalent to

$$min(max)\{v_r + ... + v_n\}$$

subject to

$$(k+1)v_0 + kv_1 + (k-1)v_2 + ... + v_k + v_{k+1} + ... + (n-k)v_n = 1$$

$$(k+1)kv_0 + (k+1)kv_1 + ... + (k+i)(k-i+1)v_i + ... + 2kv_k + 2(k+1)v_{k+1} + ... + (n-k)(n+k+1)v_n = 2S_1$$

$$(k+1)k(k-1)(v_0+v_1+v_2) + ... + [(k^3-k) - (i-2)(i-1)i]v_i + ... + 3k(k-1)v_k$$

$$+3(k+1)kv_{k+1} + ... + (n-k)(n^2+nk+k^2-1)v_n = 6S_2$$

$$v \ge 0 \ . \tag{4.38}$$

Let A be the coefficient matrix of the equality constraints in (4.25). By Theorem 2, a dual feasible basis for minimization problem in (4.25) is of the form

$$B_{min} = \begin{cases} (a_{r-1}, a_r, a_{r+1}) & \text{if } 1 \le r \le n-1 \\ (a_0, a_1, a_n) & \text{if } r = 0 \\ (a_0, a_{n-1}, a_n) & \text{if } r = n \end{cases}$$

and a dual feasible basis for maximization problem in (3.29) is of the form and

$$B_{max} = \begin{cases} (a_0, a_r, a_n) & \text{if } 1 \le r \le n-1 \\ (a_r, a_j, a_{j+1}) & \text{if } 0 \le r \le n \end{cases} ,$$

where $1 \leq j \leq n-1$.

Optimality Conditions for B_{min}

First we remark that the basis $B_{min} = (a_0, a_1, a_n),\ r = 0$ is ruled out because it contradicts with the assumption that $r \geq k+1$.

The basis $B_{min} = (a_{r-1}, a_r, a_{r+1})$ is also primal feasible if the following conditions are satisfied:

Case 2.1 If $r = k+1$, then

$$\begin{aligned} 2(k^2+5k+2)S_1 - 4(k+3)S_2 &\leq k(k+1)(k+3) \\ 4kS_1 - 4S_2 &\geq k(3k+1) \\ 2(k^2+k+2)S_1 - 4(k+1)S_2 &\geq k(k-1)^2\ . \end{aligned} \tag{4.39}$$

If (4.34) is satisfied, then the lower bound for $P(\xi = r)$ is the following:

$$\frac{k(3k+1) - 4kS_1 + 4S_2}{(k-1)(k+2)}\ . \tag{4.40}$$

Case 2.2 If $k+2 \leq r \leq n-1$, then

$$\begin{aligned} 2(2r+k)S_1 - 6S_2 &\geq r(r+2k+2) + k \\ 2(2r+k-1)S_1 - 6S_2 &\leq r(r+2k+1) \\ 2(2r+k+1)S_1 - 6S_2 &\leq (r+1)(r+2k+2)\ . \end{aligned} \tag{4.41}$$

In this case the lower bound for $P(\xi = r)$ can be given as follows:

$$\begin{aligned} &\frac{2(2r^2 - k^2 - rk + 5r + k)S_1 - 6(r+k-2)S_2}{2(r-k)(r-k+1)} \\ &+ \frac{r(r+2k+1)}{2(r-k+1)} - \frac{r(r+2k+2)+k}{r-k}\ . \end{aligned} \tag{4.42}$$

The basis $B_{min} = (a_0, a_{n-1}, a_n)$ is also primal feasible if

$$\begin{aligned} 2(n+k-1)S_1 - 6S_2 &\leq nk \\ 2(n+k)S_1 - 6S_2 &\geq (n+1)k \\ 2(k-1)S_1 - 6S_2 &\leq n(n+1)\ . \end{aligned} \tag{4.43}$$

If (4.38) is satisfied, the lower bound for $P(\xi = r)$ is the following:

$$\frac{2(n+k)S_1 - 6S_2}{(n-k)(n+1)} - \frac{k}{n-k} \ . \tag{4.44}$$

Optimality Conditions for B_{max}

Since dual feasible bases of the maximization problem in (4.40) are of the same type as the dual feasible bases of the maximization problem in (3.45), we give only the upper bound formulas for $P(\xi = r)$.

If $B_{max} = (a_0, a_r, a_n)$ is also primal feasible, i.e., (3.50) is satisfied, then the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{rk}{(n-r)(n-k)(n+1)} - \frac{nk}{(n-r)(r-k)(r+1)}$$
$$\frac{2(n^2+r^2+k^2+nr+k-1)S_1 - 6(n+r-k+1)S_2}{(n-k)(n+1)(r-k)(r+1)} \ . \tag{4.45}$$

If $B_{max} = (a_r, a_j, a_{j+1}),\ \ 0 \le j \le k-1$ is also primal feasible, i.e., (3.52) is satisfied, then the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{j(j+2k+1) - 2(2j+k-1)S_1 + 6S_2}{(r-k)(r-j)(r-j+1)} \ . \tag{4.46}$$

If $B_{max} = (a_r, a_j, a_{j+1}),\ \ k+1 \le j \le r-2$ is also primal feasible, i.e., (3.54) is satisfied, then the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{(j+1)(j+2k+2) - 2(2j+k+1)S_1 + 6S_2}{(r-k)(r-j)(r-j-1)} \ . \tag{4.47}$$

If $B_{max} = (a_r, a_j, a_{j+1}),\ \ r+1 \le j \le n-1$ is also primal feasible, i.e., (3.56) is satisfied, then the upper bound for $P(\xi = r)$ is given by the following formula:

$$\frac{1}{r-k} - \frac{(2j-k+2)(r+k+1)+rk}{(r-k)(j-k)(j-k+1)}$$
$$- \frac{2(2j+r+1)S_1 - 6S_2}{(r-k)(j-k)(j-k+1)} \ . \tag{4.48}$$

5 Examples

We present four examples to show that if the shape of the distribution is given, then by the use of our bounding methodology, we can obtain tighter bounds for $P(\xi \ge r)$ and $P(\xi = r)$.

Example 1.

We assume that the probabilities $p_0, ..., p_n$ form a unimodal sequence. Let $n = 10$, $k = 6$, $r = 4$, $S_1 = 5.556$ and $S_2 = 16.779$.

First we consider the following binomial moment problem, where the shape information is not given.

$$min(max) \sum_{i=4}^{10} p_i$$

subject to

$$\sum_{i=0}^{10} p_i = 1$$

$$\sum_{i=1}^{10} ip_i = 5.556$$

$$\sum_{i=2}^{10} \binom{i}{2} p_i = 16.779$$

$$p_0, ..., p_{10} \geq 0 \ . \tag{5.1}$$

The optimum values of problem (5.1) provides us with the following lower and upper bounds:

$$0.445 \leq \ P(\xi \geq 4) \leq 0.967 \ . \tag{5.2}$$

Now, we assume that the probability distribution in (5.1) is unimodal, i.e., $p_0 \leq ... \leq p_5 \leq p_6 \geq p_7 \geq ... \geq p_{10}$. In this case the lower and upper bounds are given as follows:

$$0.707 \leq P(\xi \geq 4) \leq 0.772 \ . \tag{5.3}$$

One can easily see that these bounds are the optimum values of problem (5.1) together with the shape constraint $p_0 \leq ... \leq p_5 \leq p_6 \geq p_7 \geq ... \geq p_{10}$.

Example 2.

In this example we assume that the probabilities $p_0, ..., p_n$ form a unimodal sequence and we find lower and upper bounds for $P(\xi = 4)$.

Let $n = 10$, $k = 6$, $r = 4$, $S_1 = 5.556$ and $S_2 = 16.779$.

We consider the following binomial moment problem, where the shape information is not given.

$$min(max) \ \ \{p_4\}$$

subject to

$$\sum_{i=0}^{10} p_i = 1$$

$$\sum_{i=1}^{10} i p_i = 5.556$$

$$\sum_{i=2}^{10} \binom{i}{2} p_i = 16.779$$

$$p_0, ..., p_{10} \geq 0 \ . \tag{5.4}$$

The optimum values of problem (5.4) provides us with the following lower and upper bounds:

$$0 \leq \ P(\xi = 4) \leq 0.685 \ . \tag{5.5}$$

If we add the shape constraint, $p_0 \leq ... \leq p_5 \leq p_6 \geq p_7 \geq ... \geq p_{10}$, to problem (5.4), then the lower and upper bounds can be given as follows:

$$0.061 \leq P(\xi = 4) \leq 0.128 \ . \tag{5.6}$$

Example 3.

Let $n = 10$, $k = 6$, $r = 8$, $S_1 = 5.556$ and $S_2 = 16.779$.

First we consider the following binomial moment problem, where the shape information is not given.

$$min(max) \sum_{i=8}^{10} p_i$$

subject to

$$\sum_{i=0}^{10} p_i = 1$$

$$\sum_{i=1}^{10} i p_i = 5.556$$

$$\sum_{i=2}^{10} \binom{i}{2} p_i = 16.779$$

$$p_0, ..., p_{10} \geq 0 \ . \tag{5.7}$$

The optimum values of problem (5.7) provides us with the following lower and upper bounds:

$$0.007 \leq \ P(\xi \geq 8) \leq 0.578 \ . \tag{5.8}$$

Now, we assume that the probability distribution in (5.7) is unimodal, i.e., $p_0 \leq ... \leq p_5 \leq p_6 \geq p_7 \geq ... \geq p_{10}$. In this case the lower and upper bounds are given as follows:

$$0.244 \leq P(\xi \geq 8) \leq 0.311 \ . \tag{5.9}$$

These bounds are the optimum values of problem (5.7) together with the shape constraint $p_0 \leq ... \leq p_5 \leq p_6 \geq p_7 \geq ... \geq p_{10}$.

Example 4.

In this example we assume that the probabilities $p_0, ..., p_n$ form a unimodal sequence and we find lower and upper bounds for $P(\xi = 8)$.

Let $n = 10$, $k = 6$, $r = 4$, $S_1 = 5.556$ and $S_2 = 16.779$. We consider the following binomial moment problem, where the shape information is not given.

$$min(max) \ \ \{p_8\}$$

subject to

$$\sum_{i=0}^{10} p_i = 1$$

$$\sum_{i=1}^{10} ip_i = 5.556$$

$$\sum_{i=2}^{10} \binom{i}{2} p_i = 16.779$$

$$p_0, ..., p_{10} \geq 0 \ . \tag{5.10}$$

The optimum values of problem (5.10) provides us with the following lower and upper bounds:

$$0 \leq \ P(\xi = 8) \leq 0.578 \ . \tag{5.11}$$

If we add the shape constraint, $p_0 \leq ... \leq p_5 \leq p_6 \geq p_7 \geq ... \geq p_{10}$, to problem (5.10), then the lower and upper bounds can be obtained as follows:

$$0.082 \leq P(\xi = 8) \leq 0.157 \ . \tag{5.12}$$

6 Applications

In this section we present two examples for the application of our bounding technique, where shape information about the unknown probability distribution can be used.

Application 1. *Application in Reliability*

Let $A_1, ..., A_n$ be independent events and define the random variables $X_1, ..., X_n$ as the characteristic variables corresponding to the above events, respectively, i.e.,

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Let $p_i = P(X_i = 1),\ i = 1, ..., n.$ The random variables $X_1, ..., X_n$ have logconcave discrete distributions on the nonnegative integers, consequently the distribution of

$$X = X_1 + ... + X_n$$

is also logconcave on the same set.

In many applications it is an important problem to compute, or at least approximate, e.g., by the use of probability bounds the probability

$$X_1 + ... + X_n \geq r, \quad 0 \leq r \leq n. \tag{6.1}$$

If $I_1, ..., I_{C(n,k)}$ designate the k-element subsets of the set $\{1, ..., n\}$ and $J_l = \{1, ..., n\} \backslash I_l$, $l = 1, ..., C(n,k)$, then we have the equation

$$P(X_1 + ... + X_n \geq r) = \sum_{k=r}^{n} \sum_{l=1}^{C(n,k)} \prod_{i \in I_l} p_i \prod_{j \in J_l} (1 - p_j), \qquad 0 < r \leq n, \tag{6.2}$$

where $C(n,k) = \left(\begin{array}{c} n \\ k \end{array} \right)$.

If n is large, then the calculation of the probabilities on the right hand side of (6.2) may be hard, even impossible. However, we can calculate lower and upper bounds for the probability on the left hand side of (6.2) by the use of the sums:

$$S_k = \sum_{1 \leq i_1 < ... < i_k \leq n} p_{i_1} ... p_{i_k} = \sum_{l=1}^{C(n,k)} \prod_{i \in I_l} p_i, \qquad k = 1, ..., m, \tag{6.3}$$

where m may be much smaller than n. Since the random variable $X_1 + ... + X_n$ has logconcave, hence unimodal distribution, we can impose the unimodality condition on the probability distribution:

$$P(X_1 + ... + X_n = k), \quad k = 0, ..., n. \tag{6.4}$$

Then we solve both the minimization and maximization problems considered in Section 3.3 to obtain the bounds for the probability (6.2). If m is small the bounds can be obtained by formulas. Note that the largest probability (6.4) corresponds to

$$k_{max} = \left\lfloor (n+1)\frac{p_1 + ... + p_n}{n} \right\rfloor .$$

Note that a formula first obtained by C. Jordan (1867) provides us with the probability (6.2), in terms of the binomial moments $S_r, ..., S_n$:

$$P(X_1 + ... + X_n \geq r) = \sum_{k=r}^{n} (-1)^{k-r} \left(\begin{array}{c} k-1 \\ r-1 \end{array} \right) S_k. \tag{6.5}$$

However, to compute the binomial moments involved may be extremely difficult, if not impossible. The advantage of our approach is that we use the first few binomial moments $S_1, ..., S_m$, where m is relatively small and we can obtain very good bounds, at least in many cases.

Application 2. *Bounding the price of an option*

We refer to the paper by Prékopa [12], where in Section 6 an option price bounding method is presented. The price of the asset is supposed to follow the multiplicative Brownian motion process:

$$S(t) = S(0)e^{\sigma B(t) + \mu t}$$

where $B(t),\ t \geq 0$ is the standard Brownian motion process, $\sigma > 0,\ \mu$ real constants and $S(0)$ is the initial price.

If t is the time now and T is the future time, X is the strike price and r is the rate of interest (assumed to be constant), then the price of the European call is

$$c = e^{-r(T-t)} E\left([S(T) - X]_+ \ \mid\ S(t) = S\right). \tag{6.6}$$

Since the process $B(t),\ t \geq 0$ has independent increments, we can write

$$c = e^{-r(T-t)} E\left(\left[S\frac{S(T)}{S(t)} - X\right]_+\right), \tag{6.7}$$

where

$$\frac{S(T)}{S(t)} = e^{\sigma(B(T)-B(t)) + \mu(T-t)}. \tag{6.8}$$

Let $Y = \sigma\left(B(T) - B(t)\right) + \mu(T-t)$. Then (6.8) can be written as

$$\begin{aligned} c &= e^{-r(T-t)} E\left([Se^Y - X]_+\right) \\ &= e^{-r(T-t)} E\left([Se^Y - X]\ \mid\ Y \geq \log\frac{X}{S}\right) P\left(Y \geq \log\frac{X}{S}\right). \end{aligned}$$

$$ \tag{6.9} $$

If we replace the probability distribution of Y by a discrete distribution with equidistant support, and assume that a few moments of Y as well as a few conditional moments of Y, given that $Y \geq \log \frac{X}{S}$, are known or can be estimated from empirical data, then we apply our bounding technique with shape constraint to obtain bounds for

$$ E\left(Se^Y - X \;\mid\; Y \geq \log \frac{X}{S}\right) \tag{6.10} $$

$$ P\left(Y \geq \log \frac{X}{S}\right) \tag{6.11} $$

The bounds for (6.10) can be obtained from our former paper Subasi, Subasi, Prékopa [13] and bounds for (6.11) can be obtained by the use of the method presented in Section 3.3 of this paper.

Mandelbrot [6] argues for the use of other symmetrical distribution for the random variable Y. Note that in order to apply our bounding methodology we do not need the exact knowledge of the distribution of Y, we only need a few of its conditional moments.

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