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ON OPTIMAL REGULATION OF A STORAGE LEVEL WITH APPLICATION TO THE WATER LEVEL REGULATION OF A LAKE

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Abstract

In [10] a system of stochastic programming models was introduced for the optimal control of a storage level. Each model in this system serves to determine the optimal policy for only one period ahead though the time horizon consists of many future periods. The optimal control thus obtained can be considered an open loop control methodology. The main purpose of this paper is to present an application by giving an optimal control method for the regulation of the water level of Lake Balaton in Hungary. By solving almost 600 stochastic programming problems we analyze what would have happened if we had controlled the water level using our method between 1922 and 1970, where one decision period is one month. The numerical results show that the proposed control methodology works quite well in this case.

1 Introduction

In [10] a system of stochastic programming models was introduced for the optimal control of a storage level. Each model in this system serves to determine the optimal policy for only one period ahead though the time horizon consists of many future periods. The optimal control thus obtained can be considered an open loop control methodolology. The main purpose of this paper is to present an application by giving an optimal control method for the regulation of the water level of Lake Balaton in Hungary. By solving almost 600 stochastic programming problems we analyze what would have happened if we had controlled the water level using our method between 1922 and 1970, where one decision period is one month. The numerical results show that the proposed control methodology works quite well in this case.

Our water input stochastic process will be assumed to be Gaussian. No time homogeneity or independence, Markovian character or whatsoever will be supposed. Also the Gaussian nature of the input process is not an essential feature of our control methodology. We refer to the paper [11] where we used a multivariate gamma distribution introduced by the authors of this paper in connection with certain reservoir system operation model. It is possible to use also here the same multigamma or some other multivariate probability distribution. The case of the multivariate normal distribution – the application of which is supported by statistics in connection with Lake Balaton – is relatively simple because of the special properties of this multivariate probability density.

2 Short description of the dynamic model system used for the control of the storage level

Taking into account that application of our control methodology which we are going to present in the further sections of this paper, we shall use the terms corresponding to the water level regulation of a lake.

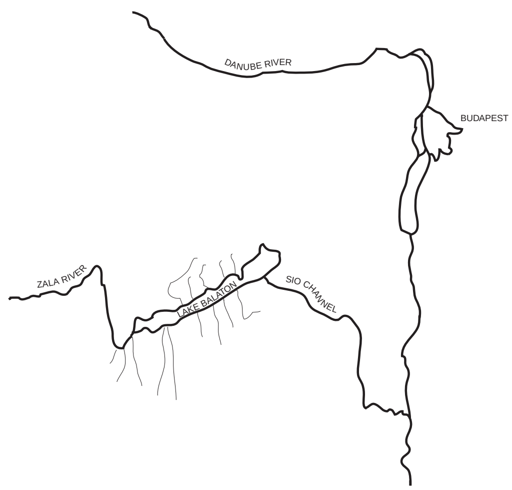

Fig. 1: Lake Balaton and catchment area in the western part of Hungary. Water is released through Sio channel into Danube river.

Fig. 1 illustrates Lake Balaton in the western part of Hungary. Small rivers and rainfall represent positive inputs whereas evaporation represents negative input. The sum of these with positive resp. negative signs will be considered the water input. Thus the variables which denote water inputs in the subsequent periods can take on negative values too. Sometimes this in fact occurs.

Water can be released through the Sio channel into the Danube river. The monthly water quantities which can be released are limited by the capacity of the Sio channel. This capacity will be denoted symbolically by K in this section.

Instead of water levels we shall speak about water quantities. The connection relative to Lake Balaton between these two notions will be clarified later on. Let ζ_0 be the initial water content of the lake and $\xi_1, \xi_2, \ldots$ the monthly random water inputs. ζ_0 will be assumed to be nonrandom in our models. Let further $z_1, z_2, \ldots$ be decision variables belonging to the subsequent periods. These are the water quantities to be released through the channel in the subsequent periods. We decide on z_1 in the beginning of the first period, on z_2 in the beginning of the second period, etc. Introduce the notations

$$\begin{aligned} \zeta_k &= \zeta_0 + \xi_1 + \cdots + \xi_k, \\ Z_k &= z_1 + \cdots + z_k, \qquad k = 1, 2, \ldots . \end{aligned}$$

The random process $\xi_1, \xi_2, \ldots$ will be assumed to be Gaussian. We prescribe further lower bounds $a_1, a_2, \ldots$ resp. upper bounds $b_1, b_2, \ldots$ for the water quantities being in the lake at the end of the subsequent periods. We consider the situation favourable in the inequalities

$$a_k \le \zeta_k - Z_k \le b_k, \qquad k = 1, 2, \ldots \tag{2.1}$$

are satisfied, where the water quantities to be released $z_1, z_2, \ldots$ are subject to the inequalities

$$0 \le z_k \le K, \qquad k = 1, 2, \ldots . \tag{2.2}$$

Since $\xi_1, \xi_2, \ldots$ are random variables, the fulfilment of the inequalities (2.1) cannot be guaranteed with probability 1. Before formulating our decision principle we make a remark concerning stochastic programming model construction.

Stochastic programming problems are formulated in such a way that first we formulate a deterministic mathematical programming problem, which is called underlying deterministic problem, then observe that some of the parameters in this problem are random in reality; in view of this, the problem looses its original meaning, hence we formulate another decision principle by taking into account the probability distribution of the random variables involved. Underlying mathematical programming problems can be either minimization (resp. maximization) problems or problems where we only wish to find at least one vector satisfying certain constraints. In this latter case the advised stochastic programming decision principle is to find that vector which maximizes the probability of the fulfilment of the random constraints subject to those constraints which do not contain random variables.

Inequalities (2.1) and (2.2) represent an underlying deterministic problem where we want to find $z_1, z_2, \ldots$ such that the referred inequalities be satisfied. The number of the considered periods should be finite. Having observed that $\xi_1, \xi_2, \ldots$ are random variables, we formulate the stochastic programming decision principle, in accordance with the above remark, so that we maximize the probability of the fulfilment of the inequalities (2.1) subject to the constraints (2.2).

Since we have the possibility to use a dynamic type decision methodology i.e. we have the possibility to decide in every period, the above mentioned principle will be turned into a sequence of problems and conditional probabilities will be maximized where in the condition there stand the already realised values of the stochastic process $\xi_1, \xi_2, \ldots$. The first problem in this sequence is the following

$$\begin{aligned} &\text{maximize} \quad P(a_k \le \zeta_k - Z_k \le b_k,\ k = 1, \ldots, N) \\ &\text{subject to} \quad 0 \le z_k \le K, \quad k = 1, \ldots, N. \end{aligned} \tag{2.3}$$

Out of the optimal solution $z_1^*, \ldots, z_N^*$ we only accept z_1^* and formulate the next problem. For the sake of simplicity the asterisk will be omitted except for Section 5. Assume that we already fixed $z_1, \dots, z_n$. Then in order to fix z_{n+1}, we formulate the following nonlinear programming problem

$$\begin{aligned} &\text{maximize} \quad P(a_k \leq \zeta_k - Z_k \leq b_k,\ k = n+1, \dots, n+N \mid \xi_1, \dots, \xi_n) \\ &\text{subject to} \quad 0 \leq z_k \leq K, \quad k = n+1, \dots, n+N. \end{aligned} \tag{2.4}$$

Here $z_{n+1}, \dots, z_{n+N}$ are the decision variables. Having computed the optimal solution, we only accept z_{n+1} as a final value. Thus our control methodology is fixed. It should be mentioned that a positive lower bound for z_k may be required. Mathematically this does not present any difficulty. In fact if K_0 is a positive lower bound for the z_k, then using the new variables $y_k = z_k - K_0$, we can transform our problem into the already introduced form (2.3), (2.4).

3 Mathematical properties of the model system introduced in Section 2

Before starting to discuss the subject mentioned in the title of this section, we recall some facts concerning logarithmic concave measures.

A nonnegative function $f(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^m$ is said to be a logarithmic concave (point) function if for every $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^m$ and $0 < \lambda < 1$, we have

$$f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \geq [f(\boldsymbol{x}_1)]^\lambda [f(\boldsymbol{x}_2)]^{1-\lambda}.$$

A measure P defined on the measurable subsets of the space $\mathbb{R}^m$ is said to be logarithmic concave if for every pair A, B of convex subsets of $\mathbb{R}^m$ and $0 < \lambda < 1$, we have

$$P(\lambda A + (1-\lambda)B) \geq [P(A)]^\lambda [P(B)]^{1-\lambda}.$$

Here the sign $+$ denotes Minkowski addition i.e. in connection with two sets $D, G, D+G = \{\boldsymbol{d} + \boldsymbol{g} : \boldsymbol{d} \in D,\ \boldsymbol{g} \in G\}$, further the constant multiple λG of the set G is defined by the equality $\lambda G = \{\lambda \boldsymbol{g} : \boldsymbol{g} \in G\}$.

In [8, 9] the following theorem was proved.

THEOREM 1. *If a probability measure P is generated by a logarithmic concave probability density i.e. for every measurable set $C \subset \mathbb{R}^m$ we have*

$$P(C) = \int_C f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

then P is a logarithmic concave measure.

Theorem 1 implies that if A is a convex subset of $\mathbb{R}^m$, then

$$\int_{A+\boldsymbol{x}} f(\boldsymbol{t})\, \mathrm{d}\boldsymbol{t}$$

is a logarithmic concave function of the variable $\boldsymbol{x}$. This implies further that if G is an $n \times N$ matrix and $\boldsymbol{z}$ is an N-component vector, then

$$\int_{A+G\boldsymbol{z}} f(\boldsymbol{t})\, \mathrm{d}\boldsymbol{t}$$

is a logarithmic concave function of the variable $\boldsymbol{z}$.

Consider now the random vector of components $\xi_1, \ldots, \xi_{n+N}$, denote by $\boldsymbol{e}$ its expectation vector and by C its covariance matrix. By assumption this random vector has a normal distribution. Assume also that this distribution is nondegenerated. Then the probability density of this random vector exists and it is given by

$$f(\boldsymbol{x}) = \left(\frac{\det C^{-1}}{(2\pi)^n}\right)^{1/2} \mathrm{e}^{-(\boldsymbol{x}-\boldsymbol{e})'C^{-1}(\boldsymbol{x}-\boldsymbol{e})/2}, \qquad \boldsymbol{x} \in \mathbb{R}^{n+N}.$$

Out of the components $\xi_1, \ldots, \xi_{n+N}$ we form two random vectors $\boldsymbol{\xi}^{\mathrm{P}}$, $\boldsymbol{\xi}^{\mathrm{F}}$ which are the following

$$\boldsymbol{\xi}^{\mathrm{P}} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \qquad \boldsymbol{\xi}^{\mathrm{F}} = \begin{pmatrix} \xi_{n+1} \\ \vdots \\ \xi_{n+N} \end{pmatrix} \tag{3.1}$$

and partition $\boldsymbol{e}$, $\boldsymbol{x}$ accordingly. The obtained parts will be denoted by $\boldsymbol{e}^{\mathrm{P}}$, $\boldsymbol{e}^{\mathrm{F}}$ resp. $\boldsymbol{x}^{\mathrm{P}}$, $\boldsymbol{x}^{\mathrm{F}}$.

Thus we have

$$\boldsymbol{e}^{\mathrm{P}} = E(\boldsymbol{\xi}^{\mathrm{P}}) = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}, \qquad \boldsymbol{e}^{\mathrm{F}} = E(\boldsymbol{\xi}^{\mathrm{F}}) = \begin{pmatrix} e_{n+1} \\ \vdots \\ e_{n+N} \end{pmatrix}. \tag{3.2}$$

The superscripts P and F are initials of the words "Past" resp. "Future". Let us rearrange and then partition the covariance matrix C so that we obtain the following

$$\begin{pmatrix} c_{n+1,n+1} & \cdots & c_{n+1,n+N} & c_{n+1,1} & \cdots & c_{n+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n+N,n+1} & \cdots & c_{n+N,n+N} & c_{n+N,1} & \cdots & c_{n+N,n} \\ c_{1,n+1} & \cdots & c_{1,n+N} & c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n,n+1} & \cdots & c_{n,n+N} & c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} E[(\boldsymbol{\xi}^{\mathrm{F}}-\boldsymbol{e}^{\mathrm{F}})(\boldsymbol{\xi}^{\mathrm{F}}-\boldsymbol{e}^{\mathrm{F}})'] & E[(\boldsymbol{\xi}^{\mathrm{F}}-\boldsymbol{e}^{\mathrm{F}})(\boldsymbol{\xi}^{\mathrm{P}}-\boldsymbol{e}^{\mathrm{P}})'] \\ E[(\boldsymbol{\xi}^{\mathrm{P}}-\boldsymbol{e}^{\mathrm{P}})(\boldsymbol{\xi}^{\mathrm{F}}-\boldsymbol{e}^{\mathrm{F}})'] & E[(\boldsymbol{\xi}^{\mathrm{P}}-\boldsymbol{e}^{\mathrm{P}})(\boldsymbol{\xi}^{\mathrm{P}}-\boldsymbol{e}^{\mathrm{P}})'] \end{pmatrix} = \begin{pmatrix} S & U \\ U' & T \end{pmatrix}. \tag{3.3}$$

It is known that the probability distribution of the random vector $\boldsymbol{\xi}^{\mathrm{F}}$ given $\boldsymbol{\xi}^{\mathrm{P}} = \boldsymbol{x}^{\mathrm{P}}$ is a normal distribution with expectation vector

$$\boldsymbol{e}^{\mathrm{C}} = \boldsymbol{e}^{\mathrm{F}} + UT^{-1}(\boldsymbol{x}^{\mathrm{P}} - \boldsymbol{e}^{\mathrm{P}}) \tag{3.4}$$

and covariance matrix

$$S - UT^{-1}U', \tag{3.5}$$

where the superscripts in $\boldsymbol{e}^{\mathrm{C}}$ refers to the word "Conditional". Thus the conditional probability density of $\boldsymbol{\xi}^{\mathrm{F}}$ given $\boldsymbol{\xi}^{\mathrm{P}} = \boldsymbol{x}^{\mathrm{P}}$ is the following

$$f(\boldsymbol{x}^{\mathrm{F}} \mid \boldsymbol{x}^{\mathrm{P}}) = \left[\frac{\det(S - UT^{-1}U')}{(2\pi)^N}\right]^{1/2} \times \mathrm{e}^{-(\boldsymbol{x}^{\mathrm{F}}-\boldsymbol{e}^{\mathrm{C}})'(S-UT^{-1}U')(\boldsymbol{x}^{\mathrm{F}}-\boldsymbol{e}^{\mathrm{C}})/2}. \tag{3.6}$$

The function $f(\boldsymbol{x}^{\mathrm{F}} \mid \boldsymbol{x}^{\mathrm{P}})$ is logarithmic concave as a function of all variables in $\boldsymbol{x}^{\mathrm{F}}$ and $\boldsymbol{x}^{\mathrm{P}}$. Now we only need the fact that it is logarithmic concave in $\boldsymbol{x}^{\mathrm{F}}$ for every fixed $\boldsymbol{x}^{\mathrm{P}}$. Consider the following set in the space of the vectors $\boldsymbol{x}^{\mathrm{F}}$:

$$
\begin{aligned}
A = \{\boldsymbol{x}^{\mathrm{F}} \, : \, & a_{n+k} - \zeta_0 - x_1 - \cdots - x_n + z_1 + \cdots + z_n \\
& \leq x_{n+1} + \cdots + x_{n+k} \\
& \leq b_{n+k} - \zeta_0 - x_1 - \cdots - x_n + z_1 + \cdots + z_n, \qquad k = 1, \ldots, N\}.
\end{aligned} \tag{3.7}
$$

Then the probability in the objective function of Problem (2.4) can be expressed in the following manner:

$$
\begin{aligned}
& P(a_k \leq \zeta_k - Z_k \leq b_k, \quad k = n+1, \ldots, n+N \mid \xi_1 = x_1, \ldots, \xi_n = x_n) \\
& \quad = \int_{A(z_{n+1}, \ldots, z_{n+N})} f(\boldsymbol{x} \mid \boldsymbol{x}^{\mathrm{P}}) \, \mathrm{d}\boldsymbol{x},
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
& A(z_{n+1}, \ldots, z_{n+N}) \\
& \quad = A + \begin{pmatrix} z_{n+1} \\ z_{n+1} + z_{n+2} \\ \vdots \\ z_{n+1} + z_{n+2} + \cdots + z_{n+N} \end{pmatrix}.
\end{aligned}
$$

Theorem 1 and Relation (3.8) jointly imply

THEOREM 2. *The probability standing in* (3.8) *is a logarithmic concave function of the variables* $z_{n+1}, \ldots, z_{n+N}$.

In [1] the following theorem is proved.

THEOREM 3. *Let A be a convex set in $\mathbb{R}^m$, symmetric about the origin. Let f be a quasi-concave probability density in $\mathbb{R}^m$ with the property that $f(-\boldsymbol{x}) = f(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^m$. Then for every $\boldsymbol{y} \in \mathbb{R}^m$ and $0 \leq k \leq 1$ we have the inequality*

$$
\int_{A+k\boldsymbol{y}} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \geq \int_{A+\boldsymbol{y}} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.
$$

In other words, this theorem states that the probability of the set $A + t\boldsymbol{y}$ is a monotonically decreasing function in $[0, \infty)$ of the variable t for every fixed $\boldsymbol{y}$.

Applying the transformation $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{e}^{\mathrm{C}}$ in the integral standing on the right hand side of (3.8), we see that it equals the integral of the function $f(\boldsymbol{u} + \boldsymbol{e}^{\mathrm{C}} \mid \boldsymbol{x}^{\mathrm{P}})$ on those set of vectors $\boldsymbol{u}' = (u_{n+1}, \ldots, u_{n+N})$ that satisfy the inequalities:

$$
\begin{aligned}
& a_{n+k} - \zeta_{\text{initial}} - e^{\mathrm{C}}_{n+1} - \cdots - e^{\mathrm{C}}_{n+k} + z_{n+1} + \cdots + z_{n+k} \\
& \leq u_{n+1} + \cdots + u_{n+k} \\
& \leq b_{n+k} - \zeta_{\text{initial}} - e^{\mathrm{C}}_{n+1} - \cdots - e^{\mathrm{C}}_{n+k} + z_{n+1} + \cdots + z_{n+k}, \quad k = 1, \ldots, N,
\end{aligned} \tag{3.9}
$$

where

$$
\zeta_{\text{initial}} = \zeta_0 + x_1 + \cdots + x_n - z_1 - \cdots - z_n
$$

is the water content of the lake beginning of Period $n+1$. This set is symmetric about the origin if in every inequality the value standing on the right hand side is the negative of that standing on the left hand side. This is attained if and only if $z_{n+1}, \ldots, z_{n+N}$ satisfy the following equations

$$\frac{a_{n+k} + b_{n+k}}{2} - \zeta_{\text{initial}} - e^{\mathrm{C}}_{n+1} - \cdots - e^{\mathrm{C}}_{n+k} + z_{n+1} + \cdots + z_{n+k} = 0, \quad k = 1, \ldots, N. \tag{3.10}$$

In case of an arbitrary system of values $z_{n+1}, \ldots, z_{n+N}$ the set (3.9) is a shift of that set (3.9) where we use the $z_{n+1}, \ldots, z_{n+N}$ that satisfy (3.10). Taking into account that $f(\boldsymbol{u} + \boldsymbol{e}^{\mathrm{C}} \mid \boldsymbol{x}^{\mathrm{P}})$ satisfies the condition (with respect to f) of Theorem 3, we see that the probability (3.8) is maximized (unconstrained) if $z_{n+1}, \ldots, z_{n+N}$ satisfy (3.10).

4 Solution of the problem formulated in Section 2

We shall consider Problem (2.4). Problem (2.3) is very similar and does not need a separate treatment. First we show that the optimization of the function (3.8) on the cube $0 \le z_k \le K$, $k = n+1, \ldots, n+N$ can be reduced to maximizations of the same functions on at most N faces of this cube. Sometimes the constrained optimal solution can be obtained directly without any computation. In fact first we solve the system of eqs. (3.10) with respect to $z_{n+1}, \ldots, z_{n+N}$. If we have $0 \le z_k \le K$ for $k = n+1, \ldots, n+N$, then this is the optimal solution also to the constrained problem (2.4). On the other hand, if for some k we have $z_k < 0$ or for some i we have $z_i > K$, then by Theorem 3 the optimum is attained on one of those faces of the cube which can be "seen" from the point with coordinates $z_{n+1}, \ldots, z_{n+N}$. These faces can be generated as follows. If $z_k < 0$, then we adopt the face

$$z_k = 0, \qquad 0 \le z_j \le K, \qquad j = n+1, \ldots, k-1,\ k+1, \ldots, n+N;$$

if $z_i > K$, then we adopt the face

$$z_i = K, \qquad 0 \le z_j \le K, \qquad j = n+1, \ldots, i-1,\ i+1, \ldots, n+N.$$

Obviously the number of such faces is at most N.

Example. Let $n = 2$, $N = 4$ and $z_3 > K$, $0 \le z_4 \le K$, $z_5 < 0$, $z_6 > K$. Then our face collection consists of the following three faces

$$\begin{gathered}
\{z_3, z_4, z_5, z_6 \,:\, z_3 = K,\ 0 \le z_4, z_5, z_6 \le K\}, \\
\{z_3, z_4, z_5, z_6 \,:\, z_5 = 0,\ \ 0 \le z_3, z_4, z_6 \le K\}, \\
\{z_3, z_4, z_5, z_6 \,:\, z_6 = K,\ 0 \le z_3, z_4, z_5 \le K\}.
\end{gathered}$$

When optimizing in the faces we can apply various nonlinear programming methods. We have tested on this and similar stochastic programming problems the method of feasible directions, SUMT, GRG, the flexible tolerance method and the cutting plane method.

It is worth to describe shortly the application of the SUMT interior point method. Let us assume that we want to maximize the function (3.8) on the following face

$$\{z_{n+1}, \ldots, z_{n+N} \,:\, z_{n+1} = K, \quad 0 \le z_i \le K, \quad i = n+2, \ldots, n+N\}.$$

If instead of the function (3.8) we work with its logarithm, then the penalty function is given by the following formula

$$-\log P(a_k \le \zeta_k - Z_k \le b_k,\ k = n+1, \dots, n+N \mid \xi_1, \dots, \xi_n) - r \sum_{k=n+2}^{n+N} \log z_k(1 - z_k), \tag{4.1}$$

where r is a fixed positive number and $z_{n+1} = K$ in the sums $Z_k = z_1 + \cdots + z_k$, $k = 1, \dots, n+N$. The function (4.1) is convex and this fact makes the solution of the unconstrained minimization problems relatively comfortable. Several unconstrained optimization methods can be applied here [7] and we have tested a number of them. The use of a gradient free method seems to be advisable. The gradient of the function (4.1) is expressed in [10] but the formula is sophisticated.

Function values are computed by simulation at every step using a fast random number generation technique written in COMPASS for the CDC 3300 computer [3].

If we take a decreasing sequence $r_1, r_2, \dots$ tending to zero, then the SUMT interior point method converges (the conditions are trivially satisfied in our case) in the sense that (4.1) converges to the negative logarithm of the optimum value of Problem (2.4).

If $N = 2$, then the original problem (2.4) is two-dimensional but since the faces of the rectangle

$$\{z_{n+1}, z_{n+2} \,:\, 0 \le z_{n+1}, z_{n+2} \le K\}$$

are line segments we have to optimize on at most two line segments. This is done by the use of the Fibonacci search [13].

5 Method for the regulation of the water level of Lake Balaton

Having performed a large number of computations it turned out that using only two conditioning random variables (instead of the whole past history) and optimizing for two steps ahead i.e. choosing $N = 2$, a satisfactory water level methodology can be obtained.

The lake is represented by a prism the surface of which is 600 km^2. We choose as water quantity unit that quantity which increases the water level by exactly 1 mm. (This quantity equals 600 000 m^3.) All data will be given in this unit.

According to what is said above, four random variables will be involved in every optimization problem. They belong to four consecutive months and will be denoted by $\xi_1, \xi_2, \xi_3, \xi_4$ in agreement with the earlier notations. To the earliest month corresponds ξ_1, then comes ξ_2 etc.

The prescribed lower resp. upper bounds are as follows:

	Lower bounds	Upper bounds
February–June	3100 mm	3400 mm
July–January	3000 mm	3300 mm

The originally prescribed levels communicated to us were 2900 mm resp. 3400 mm for every month.

We observed that our control methodology allowed to keep the water level between the narrower limits 3000 mm resp. 3300 mm (with a satisfactory probability). However, due to large input water quantities in the first half of the year, the corresponding limits were increased by 100 mm and this improved the controllability for the most important summer months. Thus in all cases we have to solve the following type of problem:

$$\begin{array}{ll} \text{maximize} & P\left(\begin{array}{l} a_3 \le \zeta_{\text{initial}} + \xi_3 - z_3 \le b_3 \\ a_4 \le \zeta_{\text{initial}} + \xi_3 + \xi_4 - z_3 - z_4 \le b_4 \end{array} \middle| \, \xi_1, \xi_2 \right) \\ \text{subject to} & 0 \le z_3 \le 200, \qquad 0 \le z_4 \le 200, \end{array} \tag{5.1}$$

where a_3, b_3, a_4, b_4 are chosen according to the above table of the lower resp. upper bounds. The subscripts of a_3, b_3, z_3 and a_4, b_4, z_4 are chosen in accordance with the subscripts of ξ_3 and ξ_4.

It will be still more comfortable to operate with the following transformed random variables

$$\zeta_1 = \xi_1, \quad \zeta_2 = \xi_2, \quad \zeta_3 = \xi_3, \quad \zeta_4 = \xi_3 + \xi_4. \tag{5.2}$$

The covariance matrix D of the random variables ζ_1, ζ_2, ζ_3, ζ_4 can be obtained from the covariance matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{24} \\ c_{13} & c_{23} & c_{33} & c_{34} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{pmatrix}$$

of the random variables ξ_1, ξ_2, ξ_3, ξ_4. We have

$$D = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{13} + c_{14} \\ c_{12} & c_{22} & c_{23} & c_{23} + c_{24} \\ c_{13} & c_{23} & c_{33} & c_{33} + c_{34} \\ c_{13} + c_{14} & c_{23} + c_{24} & c_{33} + c_{34} & c_{33} + c_{44} + 2c_{34} \end{pmatrix}. \tag{5.3}$$

With the aid of ζ_1, ζ_2, ζ_3, ζ_4, Problem (5.1) can be written in the following manner

$$\begin{array}{ll} \text{maximize} & P\left(\begin{array}{l} a_3 \le \zeta_{\text{initial}} + \zeta_3 - z_3 \le b_3 \\ a_4 \le \zeta_{\text{initial}} + \zeta_4 - z_3 - z_4 \le b_4 \end{array} \middle| \, \zeta_1, \zeta_2 \right) \\ \text{subject to} & 0 \le z_3 \le 200, \qquad 0 \le z_4 \le 200. \end{array} \tag{5.4}$$

Let us rearrange and then partition the covariance matrix D in a way indicated here below

$$\begin{array}{cc|c} \zeta_3 \quad \zeta_4 & \zeta_1 \quad \zeta_2 & \\ \hline S_1 & U_1 & \zeta_3 \; \zeta_4 \\ U_1' & T_1 & \zeta_1 \; \zeta_2 \end{array} \tag{5.5}$$

Since $\zeta_1 = \xi_1$, $\zeta_2 = \xi_2$ it follows that $T_1 = T$ where T is taken out from that special case of (3.3) in which $n = 2$, $N = 2$. Then we have

$$E\left[\begin{pmatrix} \zeta_3 \\ \zeta_4 \end{pmatrix} \middle| \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \right] = \begin{pmatrix} e_3 \\ e_3 + e_4 \end{pmatrix} + U_1 T_1^{-1} \left[\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} - \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right]. \qquad (5.6)$$

On the next pages we present the input data (Table 1) for the 50 years between 1921 and 1970. Taking into account a longer (less reliable but improved by hydrological considerations) time series, the Institute of Water Management of the Technical University of Budapest advised the use of a Gaussian process as the mathematical model of the water input process, after a careful statistical analysis [2].

Together with the realized values of the input process we present the corresponding expectations and standard deviations.

Then we present a part of a 24×24 correlation matrix (Table 2). Assuming that the random input of one month is stochastically independent of the inputs of such months which are farther than one year then it turns out that all nonzero correlations will be contained in a 24×24 correlation matrix. In practice, however, only correlations very near the diagonal will be needed because the others are very small. That part of the correlation matrix what we present here is larger than the necessary part but it well illustrates that dependencies exist only between very near months.

As it was mentioned in Section 1 we carried out the monthly optimizations between 1922 and 1970.

First we computed twelve D matrices and to each D we also computed the corresponding two matrices $U_1 T_1^{-1}$, $S_1 - U_1 T_1^{-1} U_1'$. These matrices are fixed, they do not depend on actual values of the input time series. Then using the actual values of ζ_1 and ζ_2 we computed all conditional expectations (5.6). Finally came the 588 optimizations (one for every month in the years 1922–1970) out of which a large number were trivial i.e., the solution of the eq. (3.10) specialized to our case ($n = 2$, $N = 2$) produced such z_3 and z_4 for which $0 \le z_3, z_4 \le 200$.

As an example we consider the problem of finding the optimal water quantity to be released in July 1953. In this case the random variables $\xi_1, \xi_2, \xi_3, \xi_4$ have the following meanings

- ξ_1 input water quantity in May 1953,
- ξ_2 input water quantity in June 1953,
- ξ_3 input water quantity in July 1953,
- ξ_4 input water quantity in August 1953.

The expectations, standard deviations and the correlation matrix can be obtained from the presented tables. They are reproduced here:

$$E(\xi_1) = 29.78, \quad E(\xi_2) = -4.52, \quad E(\xi_3) = -43.44 \quad E(\xi_4) = -38.30,$$
$$D(\xi_1) = 63.11, \quad D(\xi_2) = 73.98, \quad D(\xi_3) = 73.96, \quad D(\xi_4) = 69.58,$$

$$R = \begin{array}{c} \begin{array}{cccc} \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{array} \\ \begin{pmatrix} 1.000 & 0.333 & 0.198 & 0.201 \\ 0.333 & 1.000 & 0.579 & 0.263 \\ 0.198 & 0.579 & 1.000 & 0.352 \\ 0.201 & 0.263 & 0.352 & 1.000 \end{pmatrix} \end{array} \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{array}$$

Table 1: Natural water content changes of Lake Balaton (rainfall + inflow – evaporation)

	Jan.	Feb.	March	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1921	62	133	16	26	43	−35	−102	−143	−68	4	77	58
1922	102	119	128	192	−3	−52	−103	−18	118	198	96	72
1923	85	122	239	94	−17	9	−52	−75	−30	60	119	154
1924	38	94	262	171	104	26	−50	−7	−4	−26	10	56
1925	48	75	86	48	54	−18	−22	−56	87	−8	232	106
1926	130	165	77	23	−7	74	101	97	−11	89	173	139
1927	121	79	137	62	−25	−48	−54	32	45	−1	19	36
1928	99	126	111	31	90	−51	−97	−59	81	49	76	73
1929	95	74	170	208	38	−24	−90	−101	−76	39	159	16
1930	79	131	106	172	−51	−95	−124	−27	−21	182	178	235
1931	178	208	302	166	37	−59	−156	−32	43	16	92	37
1932	101	35	135	65	59	−95	−97	−65	−59	74	18	48
1933	51	73	92	21	41	−1	−119	−41	−8	48	240	136
1934	137	69	77	−37	−62	−19	−42	−46	27	9	103	72
1935	63	147	85	23	−21	−85	−136	−57	−13	18	31	172
1936	162	199	107	63	85	4	−90	−107	27	126	97	94
1937	105	146	343	260	−4	40	8	10	67	99	222	331
1938	214	118	71	18	79	−39	−98	58	−11	34	10	87
1939	114	77	73	−61	74	9	−137	−37	2	97	92	77
1940	68	67	339	111	96	76	22	159	247	174	243	90
1941	127	231	204	216	129	−39	−83	−35	−57	70	200	148
1942	115	215	367	307	178	−66	−57	−99	−56	−27	23	61
1943	112	179	−12	−5	−13	108	32	−106	−2	−2	148	129
1944	57	113	241	11	52	96	−32	−72	−50	146	225	254
1945	191	309	215	35	−37	−79	−68	−99	−28	19	99	97
1946	68	118	60	−41	−75	−51	−63	−93	−52	7	86	84
1947	87	182	448	117	−17	−54	−77	−148	−85	−5	22	88
1948	110	106	8	85	−19	17	136	−41	−30	46	47	37
1949	85	9	−16	4	41	−73	−77	−75	−84	−2	175	52
1950	115	188	63	63	−33	−155	−90	−58	−1	67	216	190
1951	139	154	206	−27	112	167	39	−35	18	−36	21	85
1952	114	155	166	37	−49	−38	−153	−128	−26	133	97	155
1953	110	71	18	−5	40	22	−93	−51	−59	8	−22	17
1954	73	56	192	28	175	16	22	−54	−31	21	76	124
1955	121	134	152	66	10	−46	39	81	26	140	185	77
1956	88	108	160	114	63	14	−3	−75	−85	22	54	114
1957	46	293	75	17	12	−56	19	−68	14	−26	78	36
1958	99	89	47	24	−60	89	−29	−83	−5	−19	32	85
1959	98	40	11	59	9	82	63	−55	−69	−40	46	140
1960	127	147	67	63	33	−80	40	−57	−7	141	154	207
1961	121	113	39	13	50	8	−80	−110	−70	−11	90	51
1962	116	70	185	95	−20	−50	21	−115	−42	−1	220	149
1963	191	130	368	174	9	−17	−127	7	96	75	71	109
1964	42	73	250	123	77	−2	−30	−31	−2	170	82	190
1965	164	111	131	190	193	244	145	131	72	15	187	448
1966	122	202	151	146	46	38	89	87	48	19	240	203
1967	204	184	143	141	−11	96	−55	−107	60	17	33	40
1968	104	85	50	10	−65	−100	−140	70	21	40	167	69
1969	139	313	234	48	14	84	−58	−30	11	20	72	110
1970	111	182	382	223	35	−18	−64	49	−5	12	62	85
Expectations	108.96	132.34	151.22	79.74	29.78	−4.52	−43.44	−38.30	−0.74	46.00	109.46	114.46
Standard-deviations	41.52	67.04	112.84	83.51	63.11	73.98	73.96	69.58	61.95	62.51	75.15	80.60

Table 2: Correlation of natural water contents changes of Lake Balaton

	Jan.	Febr.	March	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Jan.	Febr.	March	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Jan.	1.000	0.365	0.069	0.105	−0.012	0.125	−0.019	0.148	0.159	−0.054	−0.167	−0.016	−0.180											
Febr.	0.356	1.000	0.284	0.199	−0.064	−0.046	−0.018	−0.097	0.043	−0.070	−0.037	0.045	0.020	−0.003										
March	0.069	0.284	1.000	0.630	0.244	0.036	−0.090	0.139	0.220	0.128	−0.023	0.124	0.080	0.085	−0.021									
Apr.	0.105	0.199	0.630	1.000	0.284	−0.017	0.047	0.193	0.167	0.159	0.106	0.247	0.158	0.079	−0.117	0.227								
May	−0.012	−0.064	0.244	0.284	1.000	0.333	0.198	0.201	0.074	−0.054	−0.028	0.141	0.006	0.114	−0.034	0.228	0.140							
June	0.125	−0.046	0.036	−0.017	0.333	1.000	0.579	0.263	0.223	−0.152	0.016	0.338	0.028	0.041	0.016	0.106	−0.071	−0.083						
July	−0.019	−0.018	−0.090	0.047	0.198	0.579	1.000	0.352	0.120	−0.122	0.217	0.355	0.172	0.054	−0.190	−0.031	−0.091	0.019	0.174					
Aug.	0.148	−0.097	0.139	0.193	0.201	0.263	0.352	1.000	0.597	0.260	0.301	0.287	0.194	0.201	0.076	0.097	0.311	0.084	−0.034	0.150				
Sept.	0.159	0.043	0.220	0.167	0.074	0.223	0.120	0.597	1.000	0.345	0.276	0.132	0.093	0.017	0.100	0.154	0.165	0.152	−0.078	0.206	−0.046			
Oct.	−0.054	−0.070	0.128	0.159	−0.054	−0.152	−0.122	0.260	0.345	1.000	0.387	0.308	0.204	0.112	0.255	0.270	0.321	0.127	−0.094	0.180	−0.020	−0.093		
Nov.	−0.167	−0.037	−0.023	0.106	−0.028	0.016	0.217	0.301	0.276	0.387	1.000	0.513	0.526	0.393	0.262	0.216	−0.039	0.035	−0.053	0.136	−0.000	−0.114	−0.113	
Dec	−0.016	0.045	0.124	0.247	0.141	0.338	0.355	0.287	0.132	0.308	0.513	1.000	0.559	0.349	0.108	0.101	0.187	0.195	0.163	0.257	0.115	−0.174	−0.100	0.037
Jan.	−0.180	0.020	0.080	0.158	0.006	0.028	0.172	0.194	0.093	0.204	0.526	0.559	1.000	0.356	0.069	0.105	−0.012	0.125	−0.019	0.148	0.159	−0.054	−0.167	−0.016
Febr.		−0.003	0.085	0.079	0.114	0.041	0.054	0.201	0.017	0.112	0.393	0.349	0.356	1.000	0.284	0.199	−0.064	−0.046	−0.018	−0.097	0.043	−0.070	−0.037	0.045
March			−0.021	−0.117	−0.034	0.016	−0.190	0.076	0.100	0.255	0.262	0.108	0.069	0.284	1.000	0.630	0.244	0.036	−0.090	0.139	0.220	0.128	−0.028	0.124
Apr.				0.227	0.228	0.106	−0.031	0.097	0.154	0.270	0.216	0.101	0.105	0.199	0.630	1.000	0.284	−0.017	0.047	0.193	0.167	0.159	0.106	0.247
May					0.140	−0.071	−0.091	0.311	0.165	0.321	−0.039	0.187	−0.012	−0.064	0.244	0.284	1.000	0.333	0.198	0.201	0.074	−0.054	−0.028	0.141
June						−0.083	0.019	0.084	0.152	0.127	0.035	0.195	0.125	−0.046	0.036	−0.017	0.333	1.000	0.579	0.263	0.223	−0.152	0.016	0.338
July							0.174	−0.034	−0.078	−0.094	−0.053	0.163	−0.019	−0.018	−0.090	0.047	0.198	0.579	1.000	0.352	0.120	−0.122	0.217	0.355
Aug.								0.150	0.206	0.180	0.136	0.257	0.148	−0.097	0.139	0.193	0.201	0.263	0.352	1.000	0.597	0.260	0.301	0.287
Sept.									−0.046	−0.020	−0.000	0.115	0.159	0.043	0.220	0.167	0.074	0.223	0.120	0.597	1.000	0.345	0.276	0.132
Oct.										−0.093	−0.114	−0.174	−0.054	−0.070	0.128	0.159	−0.054	−0.152	−0.122	0.260	0.345	1.000	0.387	0.308
Nov.											−0.113	−0.100	−0.167	−0.037	−0.023	0.106	−0.028	0.016	0.217	0.301	0.276	0.387	1.000	0.513
Dec.												0.037	−0.016	0.045	0.124	0.247	0.141	0.338	0.355	0.287	0.132	0.308	0.513	1.000

The transformed variables (5.2) have the following expectations and covariance matrix

$$E(\zeta_1) = 29.78, \quad E(\zeta_2) = -4.52, \quad E(\zeta_3) = -43.44, \quad E(\zeta_4) = -81.74,$$

$$D = \begin{array}{cc} \begin{array}{cccc} \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{array} & \\ \begin{pmatrix} 3982.87210011 & 1554.73630744 & 924.18788880 & 1806.81784260 \\ 1554.73630744 & 5473.04040002 & 3168.03370320 & 4521.83367240 \\ 924.18788880 & 3168.03370320 & 5470.08160018 & 7281.52175378 \\ 1806.81784260 & 4521.83367240 & 7281.52175378 & 13934.33830761 \end{pmatrix} & \begin{array}{c} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{array} \end{array} .$$

From here we obtain

$$S_1 = \begin{pmatrix} 5470.08160018 & 7281.52175378 \\ 7281.52175378 & 13934.33830761 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 924.18788880 & 3168.03370320 \\ 1806.81784260 & 4521.83367240 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 3982.87210011 & 1554.73630744 \\ 1554.73630744 & 5473.04040002 \end{pmatrix},$$

$$T_1^{-1} = \begin{pmatrix} 0.00028239 & -0.00008022 \\ -0.00008022 & 0.00020550 \end{pmatrix},$$

$$U_1 T_1^{-1} = \begin{pmatrix} 0.00684480 & 0.57689906 \\ 0.14748962 & 0.78430377 \end{pmatrix}.$$

The realized water input data are the following

$$\xi_1 = \zeta_1 = 40, \qquad \xi_2 = \zeta_2 = 22,$$

hence the conditional expectation equals

$$\begin{aligned} E\left[\begin{pmatrix} \zeta_3 \\ \zeta_4 \end{pmatrix} \middle| \zeta_1 = 40, \quad \zeta_2 = 22\right] &= \begin{pmatrix} -43.44 \\ -81.74 \end{pmatrix} \\ + U_1 T_1^{-1} \left[\begin{pmatrix} 40 \\ 22 \end{pmatrix} - \begin{pmatrix} 29.78 \\ -4.52 \end{pmatrix}\right] &= \begin{pmatrix} -28.07 \\ -59.43 \end{pmatrix}. \end{aligned} \tag{5.7}$$

The covariance matrix of the conditional distribution of ζ_3, ζ_4 given that $\zeta_1 = 40$, $\zeta_2 = 22$ is the following

$$S_1 - U_1 T_1^{-1} U_1' = \begin{pmatrix} 3636.12006366 & 4660.51286423 \\ 4660.51286423 & 10121.36024427 \end{pmatrix}. \tag{5.8}$$

Since $\zeta_{\text{initial}} = 3205$, the optimization problem (5.4) can be written in the following manner

$$\begin{aligned} &\text{maximize} \quad P\left(\begin{array}{c} -205 \le \zeta_3 - z_3 \le 95 \\ -205 \le \zeta_3 - z_3 - z_4 \le 95 \end{array} \middle| \zeta_1 = 40, \ \zeta_2 = 22 \right) \\ &\text{subject to} \quad 0 \le z_3 \le 200, \qquad 0 \le z_4 \le 200, \end{aligned} \tag{5.9}$$

and the above probability distribution is two-dimensional normal with expectation vector (5.7) and covariance matrix (5.8).

If we compute z_3, z_4 according to (3.10) we obtain the values

$$z_3 = 27, \qquad z_4 = -31.$$

It follows that the optimal z_4^* to Problem (5.9) equals zero and z_3^* is the optimal solution of the following one-dimensional problem

$$\begin{aligned} &\text{maximize} \quad P\left(\begin{array}{c} -205 \le \zeta_3 - z_3 \le 95 \\ -205 \le \zeta_4 - z_3 \le 95 \end{array} \middle| \, \zeta_1 = 40, \ \zeta_2 = 22 \right) \\ &\text{subject to} \quad 0 \le z_3 \le 200. \end{aligned} \tag{5.10}$$

Table 3: Numerical results. Application of the proposed control methodology for Lake Balaton between the years 1922–1970. The z_3 are the final accepted values. Water levels are computed with these. + sign (resp. – sign) means water level above 3400 mm (resp. below 2900 mm)

		Z3	Z4	Water Level	Probability
1922	January	0.	0.	2314.(–)	0.00
1922	February	0.	0.	2433.(–)	0.00
1922	March	0.	0.	2561.(–)	0.00
1922	April	0.	0.	2753.(–)	0.00
1922	May	0.	0.	2750.(–)	0.00
1922	June	0.	0.	2698.(–)	0.00
1922	July	0.	0.	2595.(–)	0.00
1922	August	0.	0.	2577.(–)	0.00
1922	September	0.	0.	2695.(–)	0.00
1922	October	0.	0.	2893.(–)	0.00
1922	November	0.	129.	2989.	74.10
1922	December	0.	84.	3061.	90.06
1923	January	9.	22.	3137.	96.20
1923	February	3.	144.	3256.	70.79
1923	March	155.	75.	3341.	61.29
1923	April	200.	54.	3235.	88.62
1923	May	21.	1.	3196.	82.43
1923	June	0.	0.	3205.	71.23
1923	July	0.	0.	3153.	84.81
1923	August	0.	0.	3078.	79.77
1923	September	0.	0.	3048.	80.39
1923	October	0.	40.	3108.	80.80
1923	November	67.	119.	3160.	77.34
1923	December	131.	112.	3182.	90.72
1924	January	151.	41.	3069.	96.20
1924	February	0.	99.	3163.	67.16
1924	March	52.	65.	3373.	61.29
1924	April	200.	95.	3344.	85.70
1924	May	141.	0.	3307.	82.43
1924	June	74.	73.	3258.	76.86
1924	July	66.	0.	3143.	85.98
1924	August	0.	0.	3136.	79.03
1924	September	3.	57.	3129.	92.30
1924	October	26.	116.	3077.	84.40

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1924	November	7.	87.	3080.	77.34
1924	December	0.	67.	3136.	90.50
1925	January	62.	0.	3122.	96.20
1925	February	0.	115.	3197.	70.34
1925	March	76.	62.	3207.	61.29
1925	April	5.	27.	3250.	88.72
1925	May	16.	0.	3288.	82.42
1925	June	47.	63.	3223.	76.86
1925	July	0.	0.	3201.	85.30
1925	August	14.	0.	3131.	81.95
1925	September	0.	6.	3218.	92.04
1925	October	138.	121.	3071.	84.40
1925	November	26.	97.	3277.	77.34
1925	December	200.	195.	3183.	77.45
1926	January	162.	63.	3151.	96.20
1926	February	40.	151.	3276.	70.79
1926	March	192.	89.	3162.	61.29
1926	April	0.	0.	3185.	86.00
1926	May	0.	0.	3178.	76.80
1926	June	0.	0.	3252.	69.81
1926	July	81.	0.	3271.	85.83
1926	August	130.	13.	3238.	81.95
1926	September	151.	58.	3076.	92.30
1926	October	0.	119.	3165.	84.23
1926	November	139.	131.	3198.	77.34
1926	December	200.	128.	3137.	90.72
1927	January	113.	53.	3146.	96.20
1927	February	37.	155.	3188.	70.79
1927	March	61.	68.	3264.	61.29
1927	April	85.	35.	3240.	88.72
1927	May	12.	0.	3203.	82.43
1927	June	0.	0.	3155.	72.52
1927	July	0.	0.	3101.	61.25
1927	August	0.	0.	3133.	63.79
1927	September	23.	70.	3155.	92.30
1927	October	70.	134.	3084.	84.40
1927	November	34.	98.	3069.	77.34
1927	December	0.	67.	3105.	90.44
1928	January	28.	0.	3176.	96.20
1928	February	40.	140.	3262.	70.79
1928	March	161.	77.	3212.	61.29
1928	April	23.	24.	3220.	88.72
1928	May	0.	0.	3310.	82.05
1928	June	87.	71.	3172.	76.86

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1928	July	0.	0.	3075.	72.15
1928	August	0.	0.	3016.	48.87
1928	September	0.	0.	3097.	52.93
1928	October	16.	120.	3130.	84.40
1928	November	107.	119.	3099.	77.34
1928	December	48.	99.	3125.	90.72
1929	January	69.	17.	3150.	96.20
1929	February	20.	145.	3204.	70.79
1929	March	78.	65.	3297.	61.29
1929	April	133.	41.	3451.(+)	88.72
1929	May	200.	45.	3299.	76.92
1929	June	25.	57.	3250.	76.86
1929	July	15.	0.	3146.	85.27
1929	August	0.	0.	3045.	75.32
1929	September	0.	0.	2969.	54.15
1929	October	0.	0.	3008.	38.95
1929	November	0.	69.	3167.	75.52
1929	December	155.	123.	3028.	90.72
1930	January	0.	5.	3107.	96.19
1930	February	0.	101.	3238.	69.87
1930	March	142.	77.	3202.	61.29
1930	April	11.	23.	3363.	88.72
1930	May	145.	0.	3167.	62.38
1930	June	0.	0.	3072.	53.06
1930	July	0.	0.	2948.	15.01
1930	August	0.	0.	2921.	2.97
1930	September	0.	0.	2900.	8.52
1930	October	0.	24.	3082.	14.85
1930	November	92.	166.	3168.	77.34
1930	December	189.	129.	3214.	90.72
1931	January	200.	80.	3192.	96.20
1931	February	122.	171.	3278.	70.79
1931	March	200.	108.	3380.	61.21
1931	April	200.	103.	3346.	79.80
1931	May	147.	0.	3235.	82.43
1931	June	0.	36.	3176.	76.62
1931	July	0.	0.	3020.	69.09
1931	August	0.	0.	2988.	17.73
1931	September	0.	0.	3031.	48.74
1931	October	0.	69.	3047.	81.97
1931	November	4.	105.	3136.	77.34
1931	December	87.	104.	3086.	90.72
1932	January	26.	15.	3161.	96.20
1932	February	26.	140.	3170.	70.79

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1932	March	23.	57.	3282.	61.29
1932	April	101.	40.	3245.	88.72
1932	May	18.	0.	3287.	82.43
1932	June	47.	64.	3145.	76.86
1932	July	0.	0.	3048.	47.34
1932	August	0.	0.	2983.	33.58
1932	September	0.	0.	2924.	26.62
1932	October	0.	8.	2998.	19.06
1932	November	0.	88.	3016.	76.22
1932	December	0.	44.	3064.	86.50
1933	January	0.	0.	3115.	96.05
1933	February	0.	108.	3188.	70.05
1933	March	66.	61.	3215.	61.29
1933	April	15.	28.	3220.	88.72
1933	May	0.	0.	3261.	81.88
1933	June	18.	60.	3242.	76.86
1933	July	23.	0.	3101.	85.52
1933	August	0.	0.	3060.	60.82
1933	September	0.	0.	3052.	85.17
1933	October	0.	60.	3100.	82.45
1933	November	58.	115.	3281.	77.34
1933	December	200.	200.	3217.	70.14
1934	January	200.	74.	3154.	96.20
1934	February	51.	155.	3172.	70.79
1934	March	39.	67.	3210.	61.29
1934	April	4.	27.	3169.	88.72
1934	May	0.	0.	3107.	71.05
1934	June	0.	0.	3088.	33.21
1934	July	0.	0.	3046.	40.89
1934	August	0.	0.	3000.	45.18
1934	September	0.	0.	3027.	42.11
1934	October	0.	56.	3036.	80.52
1934	November	0.	90.	3139.	77.25
1934	December	94.	107.	3117.	90.72
1935	January	66.	24.	3114.	96.20
1935	February	0.	117.	3261.	70.35
1935	March	174.	80.	3172.	61.29
1935	April	0.	0.	3195.	87.87
1935	May	0.	0.	3174.	76.89
1935	June	0.	0.	3089.	67.04
1935	July	0.	0.	2953.	23.66
1935	August	0.	0.	2896.(−)	3.21
1935	September	0.	0.	2883.(−)	1.42
1935	October	0.	20.	2901.	9.20

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1935	November	0.	35.	2932.	43.31
1935	December	0.	24.	3104.	50.99
1936	January	60.	22.	3205.	96.20
1936	February	118.	161.	3286.	70.79
1936	March	200.	108.	3193.	61.11
1936	April	4.	14.	3252.	88.72
1936	May	21.	0.	3315.	82.42
1936	June	86.	70.	3233.	76.86
1936	July	22.	0.	3121.	85.78
1936	August	0.	0.	3014.	70.77
1936	September	0.	0.	3041.	29.59
1936	October	0.	48.	3167.	81.86
1936	November	164.	146.	3100.	77.34
1936	December	72.	105.	3122.	90.72
1937	January	75.	26.	3152.	96.20
1937	February	30.	148.	3269.	70.79
1937	March	177.	83.	3435.(+)	61.29
1937	April	200.	147.	3495.(+)	46.19
1937	May	200.	70.	3291.	64.33
1937	June	2.	47.	3328.	76.86
1937	July	134.	0.	3203.	85.61
1937	August	32.	7.	3180.	81.95
1937	September	52.	50.	3195.	92.30
1937	October	115.	133.	3179.	84.40
1937	November	173.	137.	3228.	77.34
1937	December	200.	180.	3359.	86.97
1938	January	200.	195.	3373.	15.74
1938	February	200.	200.	3291.	42.93
1938	March	175.	85.	3187.	61.29
1938	April	0.	2.	3205.	88.36
1938	May	0.	0.	3284.	80.17
1938	June	58.	69.	3188.	76.86
1938	July	0.	0.	3090.	78.52
1938	August	0.	0.	3148.	55.80
1938	September	56.	87.	3081.	92.30
1938	October	0.	112.	3115.	84.26
1938	November	67.	109.	3057.	77.34
1938	December	0.	58.	3144.	89.97
1939	January	80.	2.	3178.	96.20
1939	February	58.	148.	3198.	70.79
1939	March	71.	67.	3200.	61.29
1939	April	0.	18.	3139.	88.57
1939	May	0.	0.	3213.	58.54
1939	June	0.	62.	3222.	76.84

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1939	July	13.	0.	3072.	85.77
1939	August	0.	0.	3035.	46.82
1939	September	0.	0.	3037.	76.64
1939	October	0.	55.	3134.	80.96
1939	November	114.	134.	3112.	77.34
1939	December	76.	104.	3113.	90.72
1940	January	61.	22.	3120.	96.20
1940	February	0.	126.	3187.	70.59
1940	March	59.	61.	3466.(+)	61.29
1940	April	200.	153.	3377.	30.64
1940	May	174.	3.	3300.	82.43
1940	June	70.	72.	3305.	76.86
1940	July	149.	0.	3179.	86.13
1940	August	15.	14.	3322.	81.95
1940	September	200.	160.	3369.	87.77
1940	October	200.	200.	3343.	36.85
1940	November	200.	200.	3386.	20.00
1940	December	200.	198.	3276.	9.88
1941	January	200.	117.	3203.	96.10
1941	February	88.	148.	3346.	70.79
1941	March	200.	137.	3350.	53.40
1941	April	200.	31.	3366.	88.68
1941	May	163.	0.	3332.	82.40
1941	June	106.	79.	3187.	76.86
1941	July	0.	0.	3104.	80.08
1941	August	0.	0.	3069.	62.99
1941	September	0.	0.	3012.	87.72
1941	October	0.	24.	3082.	69.05
1941	November	40.	122.	3242.	77.34
1941	December	200.	172.	3190.	87.28
1942	January	172.	61.	3133.	96.20
1942	February	24.	156.	3324.	70.79
1942	March	200.	127.	3491.(+)	57.56
1942	April	200.	154.	3598.(+)	12.74
1942	May	200.	67.	3576.(+)	9.21
1942	June	200.	149.	3310.	40.00
1942	July	59.	0.	3194.	85.74
1942	August	0.	0.	3095.	81.53
1942	September	0.	0.	3039.	81.06
1942	October	0.	19.	3012.	76.90
1942	November	0.	34.	3035.	73.46
1942	December	0.	47.	3096.	87.32
1943	January	29.	1.	3179.	96.20
1943	February	53.	144.	3305.	70.79

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1943	March	200.	106.	3093.	60.62
1943	April	0.	0.	3088.	38.15
1943	May	0.	0.	3075.	34.71
1943	June	0.	0.	3183.	26.88
1943	July	37.	0.	3178.	85.99
1943	August	20.	20.	3052.	81.95
1943	September	0.	0.	3050.	44.39
1943	October	0.	42.	3048.	81.86
1943	November	0.	85.	3196.	77.27
1943	December	171.	120.	3153.	90.72
1944	January	122.	45.	3088.	96.20
1944	February	0.	111.	3201.	59.45
1944	March	98.	71.	3344.	61.29
1944	April	200.	60.	3155.	88.53
1944	May	0.	0.	3207.	73.24
1944	June	0.	36.	3303.	76.49
1944	July	156.	0.	3115.	86.10
1944	August	0.	0.	3043.	77.91
1944	September	0.	0.	2993.	60.29
1944	October	0.	12.	3139.	59.20
1944	November	129.	151.	3235.	77.34
1944	December	200.	188.	3289.	83.48
1945	January	200.	177.	3280.	91.76
1945	February	200.	191.	3389.	70.61
1945	March	200.	160.	3404.(+)	35.50
1945	April	200.	69.	3239.	82.96
1945	May	16.	1.	3187.	82.43
1945	June	0.	0.	3108.	68.17
1945	July	0.	0.	3040.	31.79
1945	August	0.	0.	2941.	34.46
1945	September	0.	0.	2913.	2.81
1945	October	0.	8.	2932.	16.88
1945	November	0.	36.	3031.	56.57
1945	December	0.	94.	3128.	90.56
1946	January	9.	8.	3187.	96.13
1946	February	8.	9.	3297.	51.19
1946	March	9.	9.	3348.	54.57
1946	April	9.	9.	3298.	21.09
1946	May	9.	6.	3214.	8.79
1946	June	6.	0.	3157.	5.07
1946	July	0.	0.	3094.	11.64
1946	August	0.	0.	3001.	33.03
1946	September	0.	0.	2949.	45.03
1946	October	0.	0.	2956.	74.23

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1946	November	0.	0.	3042.	69.54
1946	December	0.	0.	3126.	75.57
1947	January	0.	0.	3213.	96.19
1947	February	0.	0.	3395.	53.64
1947	March	0.	0.	3843.(+)	54.73
1947	April	0.	0.	3960.(+)	79.00
1947	May	0.	0.	3943.(+)	32.30
1947	June	0.	0.	3889.(+)	7.49
1947	July	0.	0.	3812.(+)	12.31
1947	August	0.	65.	3664.(+)	18.64
1947	September	200.	120.	3379.	1.73
1947	October	200.	106.	3174.	83.50
1947	November	96.	92.	3099.	77.34
1947	December	12.	84.	3175.	90.72
1948	January	113.	5.	3172.	96.20
1948	February	50.	148.	3228.	70.79
1948	March	116.	74.	3120.	61.29
1948	April	0.	0.	3205.	58.97
1948	May	0.	0.	3186.	80.22
1948	June	0.	0.	3203.	68.84
1948	July	0.	0.	3339.	85.07
1948	August	200.	5.	3098.	81.95
1948	September	0.	0.	3068.	83.20
1948	October	0.	63.	3114.	83.28
1948	November	68.	113.	3093.	77.34
1948	December	27.	91.	3104.	90.72
1949	January	36.	3.	3153.	96.20
1949	February	12.	139.	3150.	70.79
1949	March	0.	45.	3134.	61.24
1949	April	0.	0.	3138.	60.86
1949	May	0.	0.	3179.	58.30
1949	June	0.	12.	3106.	74.20
1949	July	0.	0.	3029.	86.44
1949	August	0.	0.	2954.	29.03
1949	September	0.	0.	2870.(−)	8.73
1949	October	0.	12.	2868.(−)	2.69
1949	November	0.	30.	3043.	18.30
1949	December	32.	128.	3063.	90.72
1950	January	21.	39.	3157.	96.20
1950	February	30.	143.	3315.	70.79
1950	March	200.	113.	3178.	59.82
1950	April	0.	0.	3241.	87.05
1950	May	6.	0.	3202.	82.41
1950	June	0.	0.	3047.	71.63

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1950	July	0.	0.	2957.	3.19
1950	August	0.	0.	2899.(–)	4.42
1950	September	0.	0.	2898.(–)	1.31
1950	October	0.	21.	2965.	15.50
1950	November	0.	72.	3181.	73.61
1950	December	200.	141.	3171.	90.72
1951	January	164.	73.	3146.	96.20
1951	February	53.	163.	3247.	70.79
1951	March	156.	87.	3297.	61.29
1951	April	153.	36.	3117.	88.72
1951	May	0.	0.	3229.	55.99
1951	June	22.	77.	3375.	76.86
1951	July	200.	72.	3214.	79.08
1951	August	63.	31.	3116.	81.95
1951	September	0.	0.	3134.	91.64
1951	October	35.	114.	3062.	84.40
1951	November	0.	77.	3083.	77.31
1951	December	0.	75.	3168.	90.65
1952	January	105.	5.	3177.	96.20
1952	February	56.	147.	3276.	70.79
1952	March	188.	85.	3254.	61.29
1952	April	91.	29.	3200.	88.72
1952	May	0.	0.	3151.	80.94
1952	June	0.	0.	3113.	52.77
1952	July	0.	0.	2960.	45.98
1952	August	0.	0.	2832.(–)	3.99
1952	September	0.	0.	2806.(–)	0.00
1952	October	0.	0.	2939.	0.00
1952	November	0.	94.	3036.	72.80
1952	December	9.	105.	3182.	90.72
1953	January	147.	36.	3145.	96.20
1953	February	35.	157.	3181.	70.79
1953	March	51.	66.	3148.	61.29
1953	April	0.	0.	3143.	74.44
1953	May	0.	0.	3183.	61.06
1953	June	0.	15.	3205.	74.70
1953	July	2.	0.	3110.	85.70
1953	August	0.	0.	3059.	68.30
1953	September	0.	0.	3000.	81.39
1953	October	0.	16.	3008.	62.19
1953	November	0.	52.	2986.	74.67
1953	December	0.	12.	3003.	65.02
1954	January	0.	0.	3076.	75.22
1954	February	0.	78.	3132.	66.33

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1954	March	0.	59.	3324.	61.29
1954	April	170.	47.	3182.	88.72
1954	May	0.	0.	3357.	78.57
1954	June	170.	91.	3202.	76.86
1954	July	10.	0.	3214.	86.11
1954	August	46.	2.	3114.	81.95
1954	September	0.	0.	3083.	90.38
1954	October	0.	72.	3104.	83.96
1954	November	48.	104.	3132.	77.34
1954	December	76.	99.	3180.	90.72
1955	January	135.	25.	3166.	96.20
1955	February	55.	153.	3245.	70.79
1955	March	146.	81.	3251.	61.29
1955	April	81.	30.	3236.	88.72
1955	May	10.	0.	3236.	82.43
1955	June	0.	30.	3190.	76.50
1955	July	10.	0.	3229.	75.02
1955	August	51.	0.	3259.	81.93
1955	September	168.	66.	3117.	92.30
1955	October	30.	142.	3227.	84.40
1955	November	200.	177.	3212.	76.88
1955	December	200.	153.	3089.	89.97
1956	January	54.	46.	3123.	96.20
1956	February	0.	137.	3231.	70.75
1956	March	122.	72.	3269.	61.29
1956	April	102.	35.	3281.	88.72
1956	May	62.	0.	3282.	82.42
1956	June	38.	54.	3258.	76.86
1956	July	52.	0.	3203.	85.75
1956	August	28.	2.	3100.	81.95
1956	September	0.	0.	3015.	85.18
1956	October	0.	8.	3037.	65.20
1956	November	0.	77.	3091.	76.88
1956	December	24.	93.	3181.	90.72
1957	January	130.	18.	3097.	96.20
1957	February	0.	110.	3390.	69.59
1957	March	200.	145.	3265.	33.55
1957	April	60.	0.	3222.	88.70
1957	May	0.	0.	3234.	81.82
1957	June	0.	34.	3178.	76.69
1957	July	0.	0.	3197.	69.06
1957	August	12.	0.	3117.	81.93
1957	September	0.	0.	3131.	89.26
1957	October	29.	105.	3076.	84.48

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1957	November	10.	87.	3144.	77.34
1957	December	81.	100.	3099.	90.72
1958	January	37.	11.	3162.	96.20
1958	February	26.	140.	3225.	70.79
1958	March	105.	69.	3160.	61.29
1958	April	0.	0.	3190.	85.50
1958	May	0.	0.	3130.	77.41
1958	June	0.	0.	3219.	41.55
1958	July	54.	0.	3136.	85.69
1958	August	0.	0.	3053.	80.72
1958	September	0.	0.	3048.	61.90
1958	October	0.	47.	3029.	81.78
1958	November	0.	58.	3061.	76.47
1958	December	0.	68.	3146.	90.26
1959	January	85.	7.	3159.	96.20
1959	February	32.	147.	3167.	70.79
1959	March	23.	58.	3155.	61.29
1959	April	0.	0.	3214.	76.17
1959	May	0.	0.	3223.	80.95
1959	June	0.	19.	3305.	76.18
1959	July	142.	0.	3225.	85.94
1959	August	74.	15.	3096.	81.95
1959	September	0.	0.	3027.	84.28
1959	October	0.	21.	2987.	72.79
1959	November	0.	16.	3033.	65.57
1959	December	0.	57.	3173.	88.28
1960	January	126.	20.	3174.	96.20
1960	February	68.	155.	3253.	70.79
1960	March	160.	85.	3160.	61.29
1960	April	0.	0.	3223.	84.94
1960	May	0.	0.	3256.	82.15
1960	June	5.	57.	3171.	76.86
1960	July	0.	0.	3211.	61.20
1960	August	25.	0.	3129.	81.90
1960	September	0.	0.	3122.	91.38
1960	October	15.	104.	3248.	84.40
1960	November	200.	187.	3202.	76.09
1960	December	200.	126.	3209.	90.70
1961	January	195.	59.	3135.	96.20
1961	February	39.	165.	3209.	70.79
1961	March	100.	76.	3149.	61.29
1961	April	0.	0.	3162.	78.30
1961	May	0.	0.	3212.	69.47
1961	June	0.	38.	3220.	76.58

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1961	July	8.	0.	3132.	85.64
1961	August	0.	0.	3022.	74.58
1961	September	0.	0.	2952.	32.90
1961	October	0.	0.	2941.	29.68
1961	November	0.	21.	3031.	51.68
1961	December	0.	85.	3082.	90.27
1962	January	25.	17.	3173.	96.20
1962	February	47.	142.	3197.	70.79
1962	March	66.	66.	3316.	61.29
1962	April	159.	44.	3251.	88.72
1962	May	33.	0.	3199.	82.43
1962	June	0.	0.	3149.	71.48
1962	July	0.	0.	3170.	58.56
1962	August	0.	0.	3055.	81.58
1962	September	0.	0.	3013.	43.63
1962	October	0.	9.	3012.	68.51
1962	November	0.	51.	3232.	75.10
1962	December	200.	170.	3181.	88.90
1963	January	167.	67.	3205.	96.20
1963	February	125.	159.	3210.	70.79
1963	March	102.	86.	3476.(+)	61.29
1963	April	200.	163.	3450.(+)	18.92
1963	May	200.	51.	3259.	80.07
1963	June	0.	38.	3242.	76.79
1963	July	8.	0.	3107.	85.13
1963	August	0.	0.	3114.	61.20
1963	September	0.	73.	3210.	92.30
1963	October	138.	137.	3147.	84.40
1963	November	137.	129.	3081.	77.34
1963	December	31.	98.	3159.	90.72
1964	January	110.	21.	3091.	96.20
1964	February	0.	103.	3164.	68.93
1964	March	42.	61.	3372.	61.29
1964	April	200.	91.	3295.	86.60
1964	May	85.	0.	3286.	82.43
1964	June	48.	67.	3237.	76.86
1964	July	20.	0.	3187.	85.71
1964	August	2.	0.	3154.	81.95
1964	September	6.	45.	3146.	92.30
1964	October	42.	111.	3274.	84.40
1964	November	200.	200.	3156.	70.11
1964	December	127.	101.	3219.	90.72
1965	January	188.	38.	3195.	96.20
1965	February	112.	164.	3194.	70.79

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1965	March	79.	79.	3246.	61.29
1965	April	66.	29.	3370.	88.72
1965	May	156.	0.	3407.(+)	82.38
1965	June	200.	103.	3451.(+)	76.82
1965	July	200.	127.	3396.	18.60
1965	August	200.	102.	3327.	76.21
1965	September	200.	110.	3199.	90.88
1965	October	129.	161.	3085.	84.40
1965	November	46.	105.	3226.	77.34
1965	December	200.	150.	3474.(+)	90.28
1966	January	200.	200.	3396.	0.00
1966	February	200.	200.	3398.	37.48
1966	March	200.	133.	3349.	45.48
1966	April	180.	20.	3314.	88.72
1966	May	98.	0.	3263.	82.41
1966	June	8.	60.	3293.	76.86
1966	July	102.	0.	3280.	85.83
1966	August	132.	6.	3234.	81.95
1966	September	143.	58.	3140.	92.30
1966	October	59.	147.	3099.	84.40
1966	November	57.	106.	3282.	77.34
1966	December	200.	200.	3285.	72.05
1967	January	200.	163.	3289.	94.38
1967	February	200.	188.	3273.	70.52
1967	March	190.	99.	3225.	61.29
1967	April	53.	21.	3314.	88.72
1967	May	96.	0.	3207.	82.41
1967	June	0.	0.	3303.	72.86
1967	July	148.	0.	3100.	85.94
1967	August	0.	0.	2993.	72.95
1967	September	0.	0.	3053.	15.55
1967	October	0.	70.	3070.	83.66
1967	November	30.	106.	3073.	77.34
1967	December	0.	83.	3113.	90.70
1968	January	43.	0.	3174.	96.20
1968	February	41.	141.	3218.	70.79
1968	March	96.	68.	3172.	61.29
1968	April	0.	0.	3182.	85.85
1968	May	0.	0.	3117.	75.28
1968	June	0.	0.	3017.	34.75
1968	July	0.	0.	2877.(−)	3.64
1968	August	0.	0.	2947.	0.00
1968	September	0.	6.	2968.	60.31
1968	October	0.	59.	3008.	62.78

Table 3: (continued)

		Z3	Z4	Water Level	Probability
1968	November	0.	87.	3175.	76.82
1968	December	167.	126.	3077.	90.72
1969	January	37.	40.	3179.	96.20
1969	February	64.	146.	3428.(+)	70.79
1969	March	200.	152.	3462.(+)	23.48
1969	April	200.	101.	3310.	56.84
1969	May	89.	2.	3234.	82.43
1969	June	0.	33.	3318.	76.56
1969	July	158.	0.	3103.	85.96
1969	August	0.	0.	3073.	72.91
1969	September	0.	0.	3084.	88.47
1969	October	0.	98.	3104.	84.30
1969	November	55.	105.	3121.	77.34
1969	December	62.	98.	3168.	90.72
1970	January	120.	22.	3160.	96.20
1970	February	42.	151.	3300.	70.79
1970	March	200.	105.	3482.(+)	60.79
1970	April	200.	158.	3505.(+)	13.90
1970	May	200.	76.	3340.	61.28
1970	June	72.	57.	3250.	76.86
1970	July	18.	0.	3168.	85.32
1970	August	0.	0.	3217.	80.32
1970	September	117.	77.	3095.	92.30
1970	October	0.	125.	3107.	84.40
1970	November	52.	101.	3117.	77.34
1970	December	52.	95.	3150.	90.72

The Fibonacci search gives in 15 steps the result

$$z_3^* = 2.$$

In higher dimensional cases the values of the objective function are determined by simulation. In the two-dimensional case numerical integration is satisfactorily effective. We use a reduction formula and then one-dimensional numerical integration. The reduction formula states that if $\varphi(x, y; r)$ is the two-dimensional normal probability density function with standard marginal distributions and r is the correlation coefficient ($|r| < 1$), then we have

$$\int_a^b \int_c^d \varphi(x, y; r)\, \mathrm{d}y\, \mathrm{d}x = \int_a^b \left[\Phi\left(\frac{d - rx}{\sqrt{1 - r^2}} \right) - \Phi\left(\frac{c - rx}{\sqrt{1 - r^2}} \right) \right] \varphi(x)\, \mathrm{d}x,$$

where $\varphi(x)$ and $\Phi(x)$ denote the one-dimensional standard normal probability density resp. distribution function. The one-dimensional numerical integration is done by the Romberg–Havie procedure [5]. The computational precision is 10^3.

In Table 3 we summarize the results of the 588 optimizations. The + resp. − signs mean that the water level is higher resp. lower than originally desired. Wee see that only

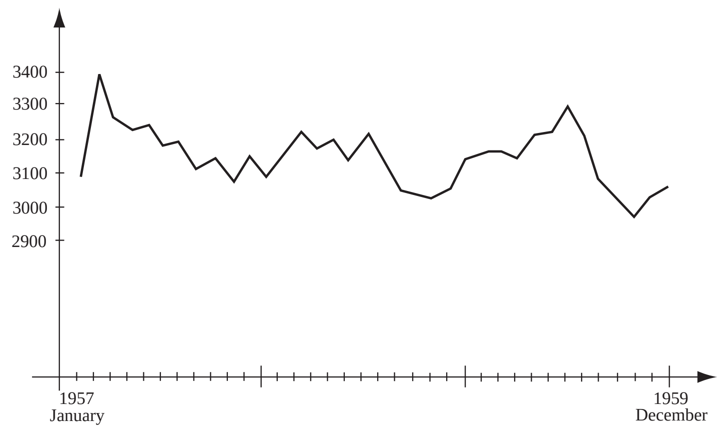

Fig. 2: Variation of the controlled water level of Lake Balaton (illustration for three years).

a few such signs occur. Moreover the first months of 1922 do not count because we need a few periods for the running-in of this control methodology. In 1946–47 the lock was repaired which caused a high water level. The variation of the controlled water level is illustrated in Fig. 2.

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