

Linear Optimization

Homework # 2 Solutions

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1)

$$C_1 = \{d \in \mathbb{R}^n \mid Ad = 0, d \geq 0\}$$

$$C_2 = \{d \in \mathbb{R}^n \mid Ad \leq 0\}$$

2) we choose 3 *linearly independent* inequalities at a time (because we are on \mathbb{R}^3) and calculate their unique solution (why unique?), an example is:

$$\begin{array}{rcccc} x_1 & +x_2 & +x_3 & = & 5 \\ x_1 & +2x_2 & & = & 6 \\ x_1 & & & = & 0 \end{array}$$

which yields $x^T = (0, 3, 2)$. We check if it satisfies the rest of the inequalities (which it does), and conclude that this is an extreme point of the feasible region (else it is not an element of this polyhedron).

3) Minkowski-Weyl representation (this is an $\{x : Ax \leq b, x \geq 0\}$ so its recession cone is $\{d : Ad \leq 0, d \geq 0\}$):

$$\{\lambda_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 5 \end{pmatrix} : \lambda_1 + \lambda_2 = 1, \lambda_1 \geq 0, \lambda_2 \geq 0\} + \{\mu_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} : \mu_1 \geq 0, \mu_2 \geq 0\}$$

The first set is the convex combination of its extreme points, the second set is the non-negative combination of its extreme directions (remember the Minkowski sum).

4) We first convert it to standard form:

$$\begin{array}{rcccccc}
 x_1 & -x_2 & +x_3 & & & & = & 1 \\
 -x_1 & +x_2 & & +x_4 & & & = & \frac{1}{2} \\
 & x_2 & & & & +x_5 & = & 1 \\
 x_1 & & & & & & \geq & 0 \\
 & x_2 & & & & & \geq & 0 \\
 & & x_3 & & & & \geq & 0 \\
 & & & x_4 & & & \geq & 0 \\
 & & & & x_5 & & \geq & 0
 \end{array}$$

Therefore the constraint matrix is:

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

A is 3×5 therefore a basis B of A should be formed of a 3 linearly independent column of A . To find all *Basic Solutions*, we must find all 3×3 submatrices of A , check if it is linearly independent, if so, it is a Basic Solution. To qualify as a Basic Feasible Solution we must further check if $B^{-1}b \geq 0$. An illustration for columns 2, 3, 5 is given below:

$$B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad (2)$$

$$B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad (3)$$

Therefore the solution defined by $x^T = (x_B^T, x_N^T) = (x_2, x_3, x_5, x_1, x_4) = ((B^{-1}b)^T, 0^T)$ is a Basic Solution.

We check $(B^{-1}b)^T = (0.5, 1.5, 0.5) \geq 0$. Therefore it is a Basic Feasible Solution.